1 Recap

- Powers of matrices are important.
- Defined eigenvalues, eigenvectors, diagonalisability, and eigenspaces. Did examples.
- Characteristic polynomial.

2 Diagonalisability

Since not every matrix is diagonalisable, it is natural to wonder when a matrix is so. Theorem: Let $u_1, \ldots, u_k$ be eigenvectors of a linear map $T : V \to V$ such that the corresponding eigenvalues are distinct. Then the eigenvectors $u_1, \ldots, u_k$ are linearly independent. As a consequence, if V is f.d. of dim n, and *all* n eigenvalues of T are distinct then T is diagonalisable. Proof of Theorem: We induct on k. For $k = 1$ it is by definition. Assume truth for $1, 2, \ldots, k-1$. Suppose $\sum_i c_i u_i = 0$. Then $\sum_i c_i T(u_i) = 0$. Hence $\sum_i c_i \lambda_i u_i = 0$. Eliminate c_1 by multiplying the first equation by λ_1 and subtracting to get $c_2(\lambda_2 - \lambda_1)u_2 + \ldots = 0$. By the induction hypothesis $c_2 = c_3 = \ldots = 0$. Thus, so is c_1. $\square$

Given two square matrices A and B when are they similar? This question is *not* easy to answer. But there are *necessary* (but NOT *sufficient*) conditions that A and B must satisfy. Assume $A = P^{-1}BP$. Then $\det(\lambda I - A) = \det(\lambda P^{-1}P - P^{-1}BP) = \det(\lambda I - B)$. So their eigenvalues must be equal! In particular, $\det(A) = \det(B)$. Moreover, we define $tr(A) = \sum_i A_{ii}$. The coefficient of λ^{n-1} is $-tr(A)$. Thus $tr(A) = \sum_i \lambda_i$. Hence $tr(A) = tr(B)$ if A and B are similar. As a part of HW you will prove that $tr(AB) = tr(BA)$.

1. Calculate the eigenvalues and eigenspaces of $T = \begin{bmatrix} 2 & 1 & 1 \\ 2 & 3 & 4 \\ -1 & -1 & -2 \end{bmatrix}$ over $\mathbb{C}$. The relevant polynomial for us here is $\det(\lambda I - T) = \begin{vmatrix} \lambda - 2 & -1 & -1 \\ -2 & \lambda - 3 & -4 \\ 1 & 1 & \lambda + 2 \end{vmatrix}$. $C_2 \to C_2 - C_3$, $R_3 \to R_3 + R_2$ and expanding along the second column yields $p_T(\lambda) = (\lambda+1)(\lambda-1)(\lambda-3)$. So the eigenvalues are $-1, 1, 3$. They are distinct and hence the matrix is diagonalisable. So we find the eigenspaces by solving $Tv = \lambda v$. For $\lambda = 1$ and $v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ we need to solve $(I - T)v = 0$. Let's do row operations to the augmented matrix $[I - T|0]$. $R_3 \to R_3 + R_1$, $R_2 \to R_2 - 2R_1$ yield $2v_3 = 0 = -2v_3 = -v_1 - v_2 - v_3$. Hence $v_3 = 0, v_2 = -v_1$. Thus every eigenvector corresponding to $\lambda = 1$ is of the form $t(1, -1, 0)$ where $t \neq 0$ is any complex number. Likewise, the eigenspace of $\lambda = -1$ is spanned by $(0, 1, -1)$ and that of $\lambda = 3$ is spanned by $(2, 3, -1)$. To find a matrix P such that $P^{-1}TP$ is $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$, note that $P^{-1}TPe_1 = De_1 = \lambda_1 e_1 = e_1$. Thus the first column of P must be an eigenvector

corresponding to $\lambda = 1$, i.e., $\begin{matrix} 1 \\ -1 \\ 0 \end{matrix}$ will do. Likewise for the other columns. Thus P is $\begin{bmatrix} 1 & 0 & 2 \\ -1 & 1 & 3 \\ 0 & -1 & -1 \end{bmatrix}$. More generally, if we have a basis of eigenvectors then $P^{-1}TP$ is diagonal where the columns of P are the eigenvectors. So $T = PDP^{-1}$.

2. $T = \begin{bmatrix} 2 & 1 & 1 \\ 2 & 3 & 2 \\ 3 & 3 & 4 \end{bmatrix}$. $\det(\lambda I - T) = (\lambda - 1)^2(\lambda - 7)$ by $C_2 \to C_2 - C_1$, $R_3 \to R_3 + R_2$ and expanding along the second column. For $\lambda = 1$, consider $[I - T|0]$ and do $R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 3R_1$ to get $-v_1 - v_2 - v_3 = 0$. Hence $(1, 0, -1), (0, 1, -1)$ span the eigenspace of $\lambda = 1$. Likewise, $(1, 2, 3)$ spans the eigenspace of $\lambda = 7$. Thus $P^{-1}TP = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 7 \end{bmatrix}$ where $P = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ -1 & -1 & 3 \end{bmatrix}$.

3. $T = \begin{bmatrix} 2 & -1 & 1 \\ 0 & 3 & -1 \\ 2 & 1 & 3 \end{bmatrix}$. $\det(\lambda I - T) = (\lambda - 2)^2(\lambda - 4)$. For $\lambda = 2$, the eigenspace is 1-dimensional and is spanned by $(-1, 1, 1)$. For $\lambda = 4$, it is spanned by $(1, -1, 1)$. Hence T is NOT diagonalisable.