1 Recap

- Gauss Jordan elimination

2 More on linear equations

In general, given an arbitrary row-reduced echelon matrix C, the number of non-zero rows is called the *row rank* of C. It is the number of pivots in C. It is also the dimension of the row space (HW). Bear in mind that the row space does not change under row operations.
Using the nullity-rank theorem one can prove that the row rank of C equals its *column rank* (the dimension of the column space/the number of non-zero columns in the Column reduced Column-echelon form/the number of non-zero rows in the RREF of A^T). Thus we can talk unambiguously about the *rank* of a matrix.
Returning back to $[\tilde{A}|\tilde{b}]$, the number of "free variables" equals the number of columns minus the row rank.

3 Inverses of linear maps vs matrices

An $n \times n$ (*square*) matrix A is said to be *invertible* if there exists an $n \times n$ matrix B such that $BA = AB = I$. B is called the *inverse* of A and is denoted as A^{-1}. Recall that A defines a linear map $T : \mathbb{F}^n \to \mathbb{F}^n$ as $T(v) = Av$ and the matrix of T in the standard basis of $\mathbb{F}^n$ is A. Note that the column space of A is the range of T.
Here is an important result: The linear map T is invertible if and only if A is an invertible matrix. Moreover, the matrix associated to T^{-1} in the standard basis of $\mathbb{F}^n$ is A^{-1}.

Proof: If T is (left-)invertible: There is a map $T^{-1} : V \to V$ such that $T^{-1}T = TT^{-1} = I$. Suppose the matrix associated to T^{-1} is B. By properties of composition, $BA = AB = I$. Hence A is invertible and B is its inverse.
If A is (left-)invertible: There is a matrix B such that $BA = I$. Hence, the corresponding linear map $\tilde{T}$ satisfies $\tilde{T}T = I$. Since a left inverse is *the* inverse, $T\tilde{T} = I$. (As a consequence, the left inverse of the *matrix* A is its right inverse.) $\square$

An $n \times n$ matrix A is invertible if and only if its (column) rank is n. (Sometimes, one says "the column rank is full" or "the matrix has full rank".) Alternatively, A is invertible if and only if $Ax = 0$ has a trivial solution.
Proof: If A is invertible: The linear map $T : \mathbb{F}^n \to \mathbb{F}^n$ defined as $T(v) = Av$ is also invertible. Hence, if $e_1, \ldots, e_n$ is a basis, then so is $T(e_1), T(e_2), \ldots$. Thus, the column space is *all* of $\mathbb{F}^n$. Hence the rank is n.
If the rank is n: The column space is all of $\mathbb{F}^n$. Hence $T(e_1), T(e_2), \ldots T(e_n)$ form a basis. Therefore T is invertible. Thus, so is A.

By nullity-rank theorem, the alternative statement is true as well. $\square$
Later on, we shall see that the rank is full if and only if a certain expression called the 'determinant' is non-zero.

4 How does one compute inverses ?

The key observation is that computing an inverse is the *same* as solving a *certain* system of linear equations. This system can of course be solved (or proved to be inconsistent) using the Gauss-Jordan algorithm.
Indeed, suppose A is invertible and $[A^{-1}]_{ij} = b_{ij}$. Then $AB = I$ is equivalent to $\sum_k a_{ik}b_{kj} = \delta_{ij}$ where $\delta_{ij} = [I]_{ij}$ (the so called Kronecker delta.) In other words, for every fixed j, we have to solve a linear system for $b_{1j}, b_{2j}, \ldots$. The other way of looking at the problem is that each *column* of B is an unknown vector x_i satisfying $Ax_i = e_i$.

Gauss-Jordan elimination to compute inverses:

- We need to form n augmented matrices $[A|e_1], [A|e_2], \ldots, [A|e_n]$.
- Unless the resulting equations are *inconsistent*, that is, A is not invertible, one can bring *all* n augmented mat rices to their RREFs simultaneously, by the *same* row operations. (Indeed, the A part is the same for all n matrices.)

So in practice, one applies row operations to $[A|I]$ to get $[I|A^{-1}]$. (After all, if the column rank is full, then the RREF *is* I.) Note that this procedue also lets us know whether A is invertible or not. On paper, if A is invertible, and we know the inverse A^{-1}, any linear system $Ax = b$ can be solved using $x = A^{-1}b$. However, in practice, computing the inverse is inefficient and subject to rounding-off errors.

Example: Determine if $A = \begin{bmatrix} 2 & 3 & 4 \\ 2 & 1 & 1 \\ -1 & 1 & 2 \end{bmatrix}$ is invertible. If so, find the inverse.

We must row-reduce $[A|I]$. $R_1 \leftrightarrow R_3$ and $R_1 \to -R_1$: $\left[\begin{array}{ccc|ccc} 1 & -1 & -2 & 0 & 0 & -1 \\ 2 & 1 & 1 & 0 & 1 & 0 \\ 2 & 3 & 4 & 1 & 0 & 0 \end{array}\right]$.

To clear the first column, $R_2 \to R_2 - 2R_1, R_3 \to R_3 - 2R_1$: $\left[\begin{array}{ccc|ccc} 1 & -1 & -2 & 0 & 0 & -1 \\ 0 & 3 & 5 & 0 & 1 & 2 \\ 0 & 5 & 8 & 1 & 0 & 2 \end{array}\right]$.

$R_2 \to R_2/3$, $R_3 \to R_3 - 5R_2$, $R_1 \to R_1 + R_2$: $\left[\begin{array}{ccc|ccc} 1 & 0 & -\frac{1}{3} & 0 & \frac{1}{3} & -\frac{1}{3} \\ 0 & 1 & \frac{5}{3} & 0 & \frac{1}{3} & \frac{2}{3} \\ 0 & 0 & -\frac{1}{3} & 1 & -\frac{5}{3} & -\frac{4}{3} \end{array}\right]$.

$R_3 \to -3R_3$, $R_2 \to R_2 - \frac{5}{3}R_3$, $R_1 \to R_1 + \frac{1}{3}R_3$: $\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 2 & 1 \\ 0 & 1 & 0 & 5 & -8 & -6 \\ 0 & 0 & 1 & -3 & 5 & 4 \end{array}\right]$.

Hence A is invertible and A^{-1} is $\begin{bmatrix} -1 & 2 & 1 \\ 5 & -8 & -6 \\ -3 & 5 & 4 \end{bmatrix}$.