1 Recap

- Matrices with distinct eigenvalues are diagonalisable.
- Examples. (If A is diagonalisable, then D consists of eigenvalues and $P^{-1}AP = D$ where the i^{th} column of P is an eigenvector of λ_i such that the chosen eigenvectors are linearly independent.)

2 Diagonalisability

Three questions:

- When are the eigenvalues of a complex matrix real?
- Just by *looking* at certain kinds of matrices can we deduce that they are diagonalisable?
- Can we diagonalise by *rotations* (linear maps that preserve lengths and can be continuously deformed from one to the other)?

The answer to all three questions is "Yes" in an important special case.

3 Hermitian linear maps and matrices

Let V be a complex inner product space. Let $T : V \to V$ be linear. If v is an eigenvector of T with eigenvalue λ, then $\lambda = \frac{\langle Tv,v\rangle}{\langle v,v\rangle}$ (by an easy calculation). Note that $\bar{\lambda} = \frac{\langle v,Tv\rangle}{\langle v,v\rangle}$. Thus λ is real if and only if $\langle v, Tv\rangle = \langle Tv, v\rangle$ for that eigenvector. Likewise, it is purely imaginary if and only if $\langle v, Tv\rangle = -\langle Tv, v\rangle$.
Def: $T : V \to V$ is called *Hermitian* if $\langle Tv, w\rangle = \langle v, Tw\rangle$ for every $v, w \in V$. It is called *skew-Hermitian* if $\langle Tv, w\rangle = -\langle v, Tw\rangle$ for every $v, w \in V$. If V is a real vector space T is called symmetric or skew-symmetric instead.
Clearly all eigenvalues of Hermitian linear maps are real, whereas they are purely imaginary for skew-Hermitian ones.

If A is an $n \times n$ complex matrix consider $T : \mathbb{C}^n \to \mathbb{C}^n$ given by $T(v) = Av$. Assume that $\mathbb{C}^n$ is endowed with the usual dot product. T is Hermitian if and only if $\langle Tv, w\rangle = (Av)^T\bar{w} = v^T A^T \bar{w}$ equals $\langle v, Tw\rangle = v^T \bar{A}\bar{w}$ for all v, w. $A^T = \bar{A}$, i.e., $\overline{A^T} = A$ Define the adjoint $A^\dagger := \overline{A^T}$. So a Hermitian *matrix* satisfies $A^\dagger = A$ and a skew-Hermitian one satisfies $A^\dagger = -A$.
Let V be a f.d. complex inner product space and $T : V \to V$ be linear. Choose an orthonormal basis. Then $\langle v, w\rangle = v^T\bar{w}$. Thus the matrix of T is Hermitian if and only if T is a Hermitian linear map and likewise for skew-Hermitian.

Examples:

- Let $V = \mathcal{C}^\infty([0, 1]; \mathbb{C})$, with $\langle f, g\rangle = \int_0^1 f(t)\bar{g}(t)dt$, and $T : V \to V$ be $T(f) = xf$. Then $\langle Tf, g\rangle = \int_0^1 xf(x)\bar{g}(x)dx = \langle f, Tg\rangle$.

- Let $V = \mathcal{C}^\infty([0,1];\mathbb{C})$, with $\langle f, g\rangle = \int_0^1 f\bar{g}dt$, and $T : V \to V$ be $T(f) = \sqrt{-1}\hbar f'$. Then $\langle Tf, g\rangle = \int_0^1 \sqrt{-1}\hbar f'\bar{g}dt = (\sqrt{-1}\hbar f\bar{g})(1) - (\sqrt{-1}\hbar f\bar{g})(0) + \int_0^1 f\overline{\sqrt{-1}\hbar g'}dt$. Thus, the map is *not* Hermitian in general.

4 Orthogonality

Theorem: Let $T : V \to V$ be a Hermitian/Skew-Hermitian linear map. Let $\lambda \neq \mu$ be distinct eigenvalues of T with eigenvectors v, w respectively. Then $\langle v, w\rangle = 0$.
Proof: $\langle v, Tw\rangle = \pm\mu\langle v, w\rangle$. But $\langle v, Tw\rangle = \pm\langle Tv, w\rangle = \pm\lambda\langle v, w\rangle$. Comparing these two, since $\lambda \neq \mu$, $\langle v, w\rangle = 0$. □
In other words, if a Hermitian linear map has distinct eigenvalues then it has an *orthonormal* basis of eigenvectors. That is, a given orthonormal basis can be "rotated" to a new orthonormal basis where the matrix of T is diagonal.

5 The Spectral Theorem

Theorem: Let V be an n-dimensional complex inner product space and $T : V \to V$ be Hermitian or skew-Hermitian. Then there exist n orthonormal eigenvectors of T forming an orthonormal basis of V.

A matrix/linear map $U : \mathbb{C}^n \to \mathbb{C}^n$ preserves the usual dot product, i.e., $\langle Uv, Uw\rangle = \langle v, w\rangle$ if and only if $U^T\bar{U} = I$, i.e., $U^\dagger U = I$ and hence $UU^\dagger = I$. Def: A complex $n \times n$ matrix U is said to be unitary if $UU^\dagger = U^\dagger U = I$, i.e, $U^{-1} = U^\dagger$. A real matrix O is said to be orthogonal if $OO^T = O^T O = I$.