1 Recap

- Hermitian and Skew-Hermitian linear maps and matrices.
- Orthogonality of eigenvectors corresponding to distinct eigenvalues for a Hermitian/Skew-Hermitian map.
- Statement of spectral theorem and definition of unitary matrices.

2 The Spectral Theorem

Theorem: Let V be an n-dimensional complex inner product space and $T : V \to V$ be Hermitian or skew-Hermitian. Then there exist n orthonormal eigenvectors of T forming an orthonormal basis of V.

Before we proceed to the proof, if V is a f.d. complex inner product space then $T : V \to V$ is said to be a unitary transformation if $\langle Tv, Tv \rangle = \langle v, v \rangle$.
Note that T is unitary if and only if there is an orthonormal basis in which $[T]$ is unitary. Here is a relevant lemma.
Lemma: If $\{e_1, e_2, \ldots, e_n\}$ is an ordered orthonormal basis of V and $\{f_1, f_2, \ldots, f_n\}$ is another ordered orthonormal basis, then the function $T(\sum_i c_i e_i) = \sum_i c_i f_i$ is a linear map and is in fact a unitary map, i.e., there is only one linear map taking e_i to f_i for all i and it is a unitary map.
Proof: Suppose $v = \sum_i v_i e_i$ and $w = \sum_i w_i e_i$, then $T(av + bw) = T(\sum_i (av_i + bw_i)e_i) = \sum_i (av_i + bw_i) f_i = a \sum_i v_i f_i + b \sum_i w_i f_i = aT(v) + bT(w)$. In fact, if we know that T is linear and $T(e_i) = f_i$, by linearity, T must be the one we just wrote. Now $\langle Tv, Tw \rangle = \langle \sum_i v_i f_i, \sum_j w_j f_j \rangle = \sum_i v_i \bar{w}_i = \langle v, w \rangle$ because f_j and e_i are o.n bases. $\square$

Corollary of Spectral Theorem: If $A = \pm A^\dagger$ then $U^\dagger A U = D$ where U is a unitary matrix.
Proof: Indeed, $Tv = Av$ has an orthonormal basis of eigenvectors. Consider the unique linear map/matrix $U : \mathbb{C}^n \to \mathbb{C}^n$ taking the usual orthonormal basis to the eigenvector one. U is a unitary matrix (by above). Hence $U^\dagger A U = U^{-1} A U = D$.

Proof of Spectral Theorem: Induct on n. For $n = 1$ it is trivial. Assume truth for $n-1$. Choose any eigenvalue λ_1 of T with a normalised eigenvector u_1 ($\|u_1\| = 1$). Then $Tu_1 = \lambda_1 u_1$. Let S be the span of u_1 and $S^\perp$ be its orthogonal complement. We first note that T takes $S^\perp$ to itself, i.e., if $s \in S^\perp$ then $T(s) \in S^\perp$. Indeed, if T is Hermitian $\langle T(s), u_1 \rangle = \langle s, Tu_1 \rangle = \bar{\lambda}_1 \langle s, u_1 \rangle = 0$. Likewise if it is skew-Hermitian.
$\dim(S^\perp) = n-1$: Extend u_1 to a basis $u_1, v_2, \ldots, v_n$ of V. Using Gram-Schmidt, convert this to an orthonormal basis $u_1, w_2, \ldots, w_n$. Let $x \in S^\perp$. Write $x = x_1 u_1 + x_2 w_2 + \ldots$. Then $0 = \langle x, u_1 \rangle = x_1$. Thus $x = x_2 w_2 + \ldots$. Hence $\dim(S^\perp) = n-1$.
Now $T : S^\perp \to S^\perp$ is a Hermitian/Skew-Hermitian linear map from an $n-1$ dimensional space to itself. By the induction hypothesis, $S^\perp$ has an orthonormal basis $u_2, \ldots, u_n$ of eigenvectors. Thus $u_1, \ldots, u_n$ form an orthonormal basis of V consisting of eigenvectors

of T. $\square$

3 An ODE to a Grecian urn

A wide variety of natural laws involve functions that satisfy differential equations. In particular, Newton's laws are $m\frac{d^2\vec{r}}{dt^2} = \vec{F}$. Equations of the form $F(t, \vec{y}, \vec{y}', \ldots, \vec{y}^{(n)}) = 0$ are called *Ordinary Differential Equations* of order n. If F is *linear* in $y, y', \ldots$ etc then the ODE is said to be linear. If $\vec{y} = 0$ is a solution of a linear ODE then it is called a *homogeneous* linear equation.
Often, ODE come with *boundary conditions* like $y(0) = 0, y'(0) = 4, y''(1) = 2$, etc. Given an ODE one can ask "Is there a solution satisfying the boundary conditions ?", "Is it unique?" "Can we write a formula for it?" and "Can we compute it using an efficient algorithm?" The answer to all these questions is NO in general. But it is YES in important cases.
The study of differential equations is akin to theology. The central questions are about existence and uniqueness.

Solve $y' = 0$ for a differentiable function y:
Before you rush to an answer, we need to know what the *domain* is. For instance, if $y : (0,1) \cup (2,3) \to \mathbb{R}$, the answer is not just "constant". It is $y = c_1$ on $(0,1)$ and $y = c_2$ on $(2,3)$! So solve $y' = 0$ on (a,b).
Proof: If $y(x_1) \neq y(x_2)$, then since y is differentiable on (x_1, x_2), and it is continuous on $[x_1, x_2]$, by LMVT $0 \neq y(x_2) - y(x_1) = y'(\theta)(x_2 - x_1) = 0$. A contradiction! Thus y is a constant. $\square$
The same result works for $\mathbb{R}$.

Suppose you leave a radioactive element (like U^{238}) to its own devices without being needy. What happens after some time?
Let $N(t)$ be the number of atoms. Then $N'(t) = kN$ is the equation obeyed by the element.
Solve $y' = ky$, $y(0) = A$ on $\mathbb{R}$:
$y(t) = Ae^{kt}$ is the answer: Let $z(t) = y(t)e^{-kt}$. Then $z'(t) = 0$ and $z(0) = A$. Thus $z(t) = A$. $\square$

Suppose you have two radioactive elements that have nothing to do with each other. Then, $N_1' = k_1 N_1, N_2' = k_2 N_2$. As before, $N_1 = A_1 e^{k_1 t}$, $N_2 = A_2 e^{k_2 t}$.

Suppose we have a two competing bacteria in a petri-dish. An oversimplified model (that I just made up and has probably nothing to do with real life) is:

$$\frac{dN_1}{dt} = c_1 N_1 - c_2 N_1 N_2$$
$$\frac{dN_2}{dt} = c_3 N_2 - c_4 N_1 N_2 \quad (1)$$

This is a *non-linear* system of ODE. To analyse what happens in a short period of time, we can "linearise" it, i.e., write $N_1 = N_0 + x$, $N_2 = M_0 + y$ and neglect second-order terms. Doing so, we get a system of the following form.
Solve $x' = ax + by, y' = cx + dy$ on $\mathbb{R}$.