1 Recap

- Row rank, column rank.
- Inverses of matrices using Gauss-Jordan.

2 Elementary row matrices

Each of the row operations is *linear*, i.e., a linear map from the row space to itself. An interesting observation is: If $C = AB$, the *rows* of C are *linear combinations* of the *rows* of B and the *columns* of C are linear combinations of the columns of A. So we expect an elementary row operation on an $m \times n$ matrix A to be equivalent to left-multiplying with an $m \times m$ matrix. Indeed, these matrices exist and are called elementary row matrices.

- $R_i \leftrightarrow R_j$: $B = E_1 A$ where E_1 is the matrix obtained by interchanging the rows of $I_{m\times m}$. It is invertible with the inverse being itself.
- $R_i \rightarrow cR_i$ ($c \neq 0$): $B = E_2 A$ where E_2 is again obtained the same way. Its i^{th} row has c instead of 1. Its inverse has $\frac{1}{c}$ instead of 1.
- $R_i \rightarrow R_i + cR_j$: $B = E_3 A$ where E_3 is obtained similarly. Its inverse is obtained by replacing c with $-c$.

So row-reduction is equivalent to left-multiplying with a product of elementary row matrices. So if row-reduction leads to I, , then $EA = I$ and hence $A^{-1} = E$.

3 Determinants - motivation and definition

Suppose we consider $ax + by = e$, $cx + dy = f$. We can easily solve to get $(ad - bc)x = ed - bf$, $(ad - bc)y = af - ce$. Thus if $ad - bc = 0$ then unless $ed - bf = 0, af - ce = 0$, we cannot solve the equations. If $ad - bc \neq 0$, we have a unique solution. By our criterion for invertibility, the coefficient matrix is invertible if and only if $ad - bc \neq 0$.
Solving the above linear system is equivalent to finding the intersection set of two lines (Actually, if $a = b = e = 0$, then it is just one line and if $a = b = e = c = d = f = 0$, it is all of $\mathbb{R}^2$!)
Either they intersect at a single point or they are parallel and do not intersect or they intersect in a line or they are all of $\mathbb{R}^2$. Indeed, if $ad - bc = 0, ed - bf = 0, af - ce = 0$, they coincide. If they intersect non-trivially the area of the "obvious" parallelogram is not zero. The (signed) area is $\vec{v} \times \vec{w} = (ad - bc)\hat{k}$.

For 3×3 systems, clearly a unique solution implies that the (signed) volume of a parallelopiped is non-zero. This volume is $(\vec{u} \times \vec{v}).\vec{w}$. (The "scalar triple product".)
By analogy, the (signed) volume in n-dimensions ought to be some complicated polynomial expression in the components. This quantity shall be called the determinant of the square matrix formed by the n vectors.

Note that the (signed) volume of n-vectors in $\mathbb{R}^n$, $v_1, \ldots, v_n$ must

- *scale* with each vector,
- be 1 for the standard basis,
- *vanish* if two vectors are equal, and
- Since the only operations in a general vector space are linear combinations, we must check how the $2,3$-dimensional volumes behave. $(\vec{v}_1 + \vec{v}_2) \times \vec{w} = \vec{v}_1 \times \vec{w} + \vec{v}_2 \times \vec{w}$ and likewise for the triple product. So we *hope* that the signed volume in higher dimensions obeys this *multi-linearity* property as well. To *prove* such a statement, we need to resort to integration in more than one variable.

Let $v_1, v_2, \ldots, v_n$ be an ordered collection of n vectors in $\mathbb{F}^n$. A function F that takes this tuple to $\mathbb{F}$ is called a *determinant* function if it satisfies the following axioms.

- Scaling: If v_k is replaced with tv_k (and the other v_is are left intact), then F gets scaled by t.
- Additivity: $F(\ldots, v_k + w, \ldots) = F(\ldots, v_k, \ldots) + F(\ldots, w, \ldots)$. A function that satisfies the first two properties is said to be *multilinear*.
- Alternating: $F(\ldots, v, \ldots, v, \ldots) = 0$.
- Normalisation: $F(e_1, \ldots, e_n) = 1$.

4 Properties of an alternating (not necessarily normalised) multilinear function

- Linearity with more than one vector: $F(\ldots, v_k + c_1 w_1 + c_2 w_2 + \ldots + c_m w_m, \ldots) = F(\ldots, v_k, \ldots) + c_1 F(\ldots, w_1, \ldots) + \ldots$ (HW).
- It vanishes if some vector is 0: $F(\ldots, 0, \ldots) = 0F(\ldots, 0, \ldots) = 0$.
- (Antisymmetry) If $v_i \leftrightarrow v_j$ F changes sign: $F(\ldots, v_i + v_j, \ldots, v_i + v_j, \ldots) = 0$ and hence $F(\ldots, v_i, \ldots, v_i + v_j, \ldots) = -F(\ldots, v_j, \ldots, v_i + v_j, \ldots)$. Thus $0 + F(\ldots, v_i, \ldots v_j, \ldots) = -F(\ldots, v_j, \ldots, v_i, \ldots) + 0$.
- If the vectors are linearly dependent then F vanishes: Suppose $\sum_i c_i v_i = 0$ with $c_k \neq 0$. Then $F = \frac{1}{c_k} F(\ldots, c_k v_k, \ldots)$ which is $\frac{1}{c_k} F(\ldots, -\sum_{i \neq k} c_i v_i) = \sum_{i \neq k} \frac{-c_i}{c_k} F(\ldots, v_i, \ldots, v_i, \ldots) = 0$.