1 Recap

- Elementary row matrices.
- Determinant definition.

2 Determinants

Uniqueness theorem: Suppose d is a determinant function and f is an alternatng multilinear function. Then $f(v_1,\ldots,v_n) = d(v_1,\ldots,v_n)f(e_1,\ldots,e_n)$. So if f is also a determinant function, then $f = d$.
Proof: Let $v_i = \sum_j c_{ij}e_j$. Then $f(\sum_{j_1} c_{1j_1}e_{j_1}, \sum_{j_2} c_{2j_2}e_{j_2},\ldots) = \sum c_{1j_1}c_{2j_2}\ldots f(e_{j_1},e_{j_2},\ldots)$. If any of the j_i coincide, that term will be 0. So we may assume that all the j_i are different, i.e., $j_1,j_2,\ldots,j_n$ is a *permutation* of $1,2,\ldots,n$.
We can prove by induction on n that *any* permutation can be obtained by a finite number of *interchanges*. Indeed, it is trivial for $n = 1$. One of j_i corresponds to n. Suppose it is j_k. Now $[1,2,\ldots,n-1] \to [j_1,\ldots,j_{k-1},j_n,j_{k+1},\ldots,j_{n-1}]$ is a permutation of $n-1$ things. By the induction hypothesis, it can be obtained using a finite number of interchanges. That is, $[1,2\ldots,n] \to [j_1,\ldots,j_{k-1},j_n,j_{k+1},\ldots,j_{n-1},j_k = n]$ can be obtained that way. Now interchange j_k with j_n to get the desired permutation.
Using the above result we see that $d(e_{j_1},\ldots,e_{j_n}) = (-1)^K d(e_1,\ldots,e_n) = (-1)^K$ and $f(e_{j_1},\ldots,e_{j_n}) = (-1)^K f(e_1,\ldots,e_n) = d(e_{j_1},\ldots,e_{j_n})f(e_1,\ldots,e_n)$. Thus $f(v_1,\ldots,v_n) = \sum c_{1j_1}\ldots d(e_{j_1},\ldots,e_{j_n})f(e_1,\ldots,e_n) = d(v_1,\ldots,v_n)f(e_1,\ldots,e_n)$.

Assuming existence: 2×2 determinants: Consider $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = a\begin{vmatrix} 1 & b \\ 0 & d \end{vmatrix} + c\begin{vmatrix} 0 & b \\ 1 & d \end{vmatrix}$ which can be column-transformed to $= ad\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} + bc\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}$ which equals $ad - bc$.

Assuming existence: Upper triangular matrices: We want to compute $\det(U)$ where $U = \begin{bmatrix} u_{11} & u_{12} & \ldots & u_{1n} \\ 0 & u_{22} & \ldots & u_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & u_{nn} \end{bmatrix}$. We claim that $\det(U) = u_{11}u_{22}\ldots$. We shall prove this claim by induction on n. For $n = 1$, we are done. Assume truth for $n-1$. By scaling, $\det(U) = u_{11}\det(U')$ where U' has e_1 in the first column. By multilinearity, i.e., column transformations, we can "clear" the first row. Now $\det(e_1,v_2,\ldots) = \sum_J c_{j_2 2}c_{j_3 3}\ldots\det(e_1,e_{j_2},\ldots)$. Suppose we need K_J interchanges of columns to permute $j_2,j_3,\ldots$ to $2,3,\ldots n-1$. Then $\det(e_1,v_2,\ldots) = \sum_J c_{j_2 2}c_{j_3 3}\ldots(-1)^{K_J}\det(e_1,e_2,\ldots)$ which is $\sum_J c_{j_2 2}\ldots(-1)^{K_J} = \det(v_2,\ldots)$. Thus $\det(U) = u_{11}\det(U'')$ where U'' is the $(n-1)\times(n-1)$ matrix obtained by deleting the first row and first column. It is upper triangular. We are done by the induction hypothesis. □

Assuming existence: A crucial property (Expansion along the first column): Note that $\det(v_1,\ldots,v_n) = \det(\sum_j c_{j1}e_j, v_2,\ldots,v_n) = \sum_j c_{j1}\det(e_j,v_2,\ldots,v_n)$.
Property: If we define $n-1$-dimensional new columns/vectors $\tilde{v}_{2,j},\tilde{v}_{3,j},\ldots$ by simply

deleting the e_j components from $v_2, \ldots$ and replacing e_{j+1} with e_j, e_{j+2} with e_{j+1} etc, then $\det(v_1, \ldots) = \sum_j c_{j1}(-1)^{j+1} \det(\tilde{v}_{2,j}, \tilde{v}_{3,j}, \ldots)$. Such an $(n-1) \times (n-1)$ determinant is called a *minor*. Note that by interchanging columns a *similar* property holds for any column (if we prove it for the first column).

Proof: Fix j and neglect the j subscript in $\tilde{v}_{2,j}, \ldots$. By interchanges, $\det(e_j, v_2, \ldots) = (-1)^{j+1} \det(v_2, \ldots, v_{j-1}, e_j, v_{j+1}, \ldots)$. Claim: $\det(v_2, \ldots, v_{j-1}, e_j, v_{j+1}, \ldots) = \det(\tilde{v}_2, \ldots, \tilde{v}_{j-1}, \tilde{v}_{j+1}, \ldots)$. This claim is enough to complete the proof. Proof of claim: Note that $\det(\tilde{v}_2, \ldots, \tilde{v}_{j-1}, \tilde{v}_{j+1}, \ldots) = \sum_I \tilde{c}_{i_1 2} \ldots \det(e_{i_1}, e_{i_2}, \ldots)$ where $i_1, i_2, \ldots$ is a permutation of $1, 2, \ldots$. If we need K interchanges to bring $i_1, \ldots,$ in ascending order, then $\det(e_{i_1}, e_{i_2}, \ldots) = (-1)^K$. If the $i_1, i_2, \ldots$ are in ascending order, then we are done with the claim and the proof of the property (why?). $\square$