1 Recap

- Uniqueness of determinants.
- Consequences of existence (determinants of 2×2 matrices, upper-triangular matrices, and expansion along a column).

2 Determinants

Existence: The construction of a determinant is done recursively/inductively. For a 1×1 matrix, define $\det(A) = a_{11}$. Assume that for any $k \times k$ matrix ($k \leq n-1$), the det function exists. It is natural to try to define by expansion along the first *column*. However, we shall try $\det(v_1, \ldots, v_n) := \sum_j A_{1j}(-1)^{j+1} M_{1j}$ where the minor M_{1j} is obtained by deleting the first row and the j^{th} column. (Expansion along the first *row*.) We simply need to check that this definition satisfies all the axioms, thus completing the induction step:

- Scaling: If $v_i \to tv_i$, for $j \neq i$ each of the A_{1j} scales by t by the induction hypothesis. For $j = i$ A_{1i} remains unchanged by c_{1i} scales by t. Thus every term in the definition scales by t.
- Linearity: If $v_i \to v_i + w$, for $j \neq i$ as before, A_{1j} is linear. For $j = i$, as before, the coefficient is linear. We are done.
- Normalisation: $\det(e_1, \ldots, e_n) = A_{11} = 1$ by the induction hypothesis.
- Alternating: It is enough to prove this property for *adjacent* columns (why?) So if $v_i = v_{i+1} = v$, any minor that contains v_i AND v_{i+1} is 0 by the induction hypothesis. The only minors that remain are M_{1i}, M_{1i+1}. So $\det(v_1, \ldots, v, v, \ldots) = (-1)^i(-A_{1i}M_{1i} + A_{1i+1}M_{1i+1})$. But $A_{1i} = A_{1i+1}$ and $M_{1i} = M_{1i+1}$. Hence we are done. $\square$

The transpose of a matrix A, i.e., A^T is obtained by interchanging the rows and the columns. It is easy to show that $(AB)^T = B^T A^T$.
For any $n \times n$ matrix A $\det(A) = \det(A^T)$. (As a consequence, *row operations* of the form $R_i \to R_i + cR_j$ keep the determinant invariant, scaling a row scales the determinant and if two *rows* are equal the determinant vanishes.)
Proof: We prove by induction on n. $n = 1$ is trivial. Assume truth for $n - 1$. For n, expand A along its first *row*: $\det(A) = \sum_j A_{1j}(-1)^{1+j} M_{1j}$. Expand A^T along its first *column*: $\det(A^T) = \sum_j (A^T)_{j1}(-1)^{1+j} M'_{j1}$. But $(A^T)_{j1} = A_{1j}$ and $M'_{j1} = M_{1j}$ by the induction hypothesis. $\square$

3 Computing determinants using the Gauss-Jordan technique

Since the RREF U of a square matrix A is upper-triangular (why?), and we can use Gauss-Jordan row operations to bring it to such a form, we can compute the determinant of the matrix easily. Each time we scale a row by a constant c_i the determinant scales and each row-exchange leads to a -1. So $\det(A) = \frac{(-1)^p \det(U)}{c_1 c_2 \dots}$.

An example: Compute $\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}$ (a Vandermonde determinant)

$R_2 \to R_2 - R_1, R_3 \to R_3 - R_1$ do not change the determinant and yield $\begin{vmatrix} 1 & x & x^2 \\ 0 & y-x & y^2-x^2 \\ 0 & z-x & z^2-x^2 \end{vmatrix}$

Scaling gives $(y-x)(z-x)\begin{vmatrix} 1 & x & x^2 \\ 0 & 1 & x+y \\ 0 & 1 & x+z \end{vmatrix}$ which is (after $R_3 \to R_3 - R_2$) $(y-x)(z-x)\begin{vmatrix} 1 & x & x^2 \\ 0 & 1 & x+y \\ 0 & 0 & z-y \end{vmatrix}$ which is upper-triangular and hence equal to $(y-x)(z-x)(z-y)$.