

HW 1

1. Let V be a real vector space and $S \subseteq V$ be a subset. Suppose $S \subseteq T \subseteq V$ where T is a subspace. Prove that $L(S) \subseteq T$ where $L(S)$ is the span of S .
2. Let $a < b$ be real numbers. Suppose $w : [a, b] \rightarrow \mathbb{R}$ is a continuous function such that $w(t) > 0$ for all $t \in [a, b]$. Verify that $\langle f, g \rangle = \int_a^b w(t)f(t)\bar{g}(t)dt$ is an inner product on the vector space of continuous complex-valued functions on $[a, b]$.
3. Let H be a symmetric positive-definite real $n \times n$ matrix. Let V be a finite-dimensional real vector space with an ordered basis $e_1, \dots, e_n$. Prove that $\langle v, w \rangle = v^T H w$ is an inner product.
4. Let V be the vector space of complex polynomials of degree ≤ 3 . Let $T : V \rightarrow V$ be the map $T(p) = p(x+1)$. Let $D : V \rightarrow V$ be the differentiation map. Consider the two ordered bases (the first one for the domain and the second one for the target) : $1, x, x^2, x^3$ and $1, x+1, x^2+2x, x^3$.
 - (a) Prove that $1, x+1, x^2+2x, x^3$ is indeed a basis.
 - (b) Prove that T is a linear map.
 - (c) What is the matrix representing $TD - D^2$ in these bases ?
 - (d) Find a basis for the null space of $TD - D^2$. What is the dimension of the null space of $TD - D^2$?
 - (e) What about the dimension of its range ?
5. Let V, W be vector spaces over the same field. Let $T : V \rightarrow W$ be a linear map. If V is infinite-dimensional, then prove that at least one of $N(T)$ or $R(T)$ is infinite-dimensional.