HW 5

1. Let $w(t) : (a, b) \to \mathbb{C}$ be a function such that $w(t) = u(t) + \sqrt{-1}v(t)$ where $u, v : (a, b) \to \mathbb{R}$ are differentiable. We then say that $w(t)$ is differentiable and $w' = u' + \sqrt{-1}v'$. Prove that if $w_1(t), w_2(t)$ are two such differentiable complex-valued functions, then so is $w_1(t)w_2(t)$ and its derivative is $w_1'w_2 + w_1w_2'$. Prove that if $w_2 \neq 0$, then $\frac{w_1}{w_2}$ is differentiable and its derivative is $\frac{w_1'w_2 - w_2'w_1}{w_2^2}$. If $g : (a, b) \to (a, b)$ is differentiable, then prove that $w(g(t))$ is also differentiable and its derivative is $w'(g(t))g'(t)$.

2. This problem tells you what to do if the matrix (of a linear system of ODE) is not diagonalisable:
   Let $P \in \mathbb{R}$ be a constant. Let $A = \begin{bmatrix} -P & \frac{-P^2}{4} \\ 1 & 0 \end{bmatrix}$.

   (a) Prove that $\lambda = -\frac{P}{2}$ is the only eigenvalue and its eigenspace is one-dimensional.

   (b) Let u be an eigenvector of $-\frac{P}{2}$ and let v be one of e_1, e_2 such that u, v form a basis. Prove that in this basis, A is upper-triangular.

   (c) Now solve $\frac{d\vec{y}}{dt} = A\vec{y}$ on $\mathbb{R}$ and prove that the solution space is a two-dimensional real vector space spanned by $e^{\lambda t}$ and $te^{\lambda t}$.

3. If $U \subset \mathbb{R}^m$ and $V \subset \mathbb{R}^n$ are open, then prove that $U \times V \subset \mathbb{R}^{n+m}$ is open.

4. In each of the following cases, prove (or disprove):

   (a) If A_i are infinitely many open sets, then their intersection $\cap_i A_i$ is open.

   (b) If A_i are infinitely many closed sets, then their intersection $\cap_i A_i$ is closed.

   (c) The set $\{(x, y, z) \in \mathbb{R}^3 \mid |x + y| < 1, |z| < 1\}$ is open.

5. Prove that a set S is closed if and only if $S = Int(S) \cup \partial S$.

6. If $\lim_{(x,y)\to(a,b)} f(x, y) = L$ exists, and $\lim_{x\to a} f(x, y), \lim_{y\to b} f(x, y)$ exist, then prove that $\lim_{x\to a} \lim_{y\to b} f(x, y) = \lim_{y\to b} \lim_{x\to a} f(x, y) = L$.

7. Let $f(x, y) = x \sin(1/y)$ if $y \neq 0$ and $f(x, 0) = 0$. Show that $f(x, y) \to 0$ as $(x, y) \to (0, 0)$ but $\lim_{x\to 0} \lim_{y\to 0} f(x, y) \neq \lim_{y\to 0} \lim_{x\to 0} f(x, y)$. Why does this not contradict the earlier result?