1 Logistics

- Office: N23, Email : vamsipingali@iisc.ac.in
- Course webpage :
  http://math.iisc.ac.in/ vamsipingali/teaching/um102anallinealg2022spring/um1022022.html
- Tests/Quizzes based on HW : 20% (Closed book.), Midterm - 30%, and Final - 50%.
- TAs: Aashirwad N. Ballal, Geethika Sebastian, Mrigendra K, Sivaram P.
- Text book: Apostol, Calculus (Vol 2).

2 Why care about Linear Algebra ?

Linear algebra originated from "word-problems" in high-school, i.e., things that lead to equations like $5x + 3y = 1, x - y = 7$. To solve them systematically, *matrices* were invented.On paper, everything in (finite-dimensional) linear algebra can be done using matrices. So why define "abstract" things such as vector spaces ? The point is that linear equations come in several guises(Numerical equations, Polynomial equations, Differential Equations, etc). All such equations rely on similar manipulations.So going by the spirit of algebra,we abstract out the essential features of such manipulations into a definition and prove general theorems about such objects. While that was the original reason to invent linear algebra,today, it goes much further.Google uses Linear Algebra for instance !

3 Vector spaces

Solving linear equations (like the word-problem equations)requires "cross-multiplication" by "numbers" and adding/subtracting equations. So any set that allows scalar-multiplication (with real or complex numbers or more general "numbers" belonging to a "field"),and addition and subtraction (that behave "well" with scalar-multiplication) should allow linear equations and their solutions by the same high-school algorithm. To this end, recall that a vector space V over a field $\mathbb{F}$ (if you find fields confusing, whenever I say $\mathbb{F}$, replace it with $\mathbb{R}$ or $\mathbb{C}$ in your minds)is a set V equipped with binary operations$+ : V \times V \to V$ and $. : \mathbb{F} \times V \to V$ satisfying a bunch of axioms:

- Commutativity of addition : $v + w = w + v$.
- Existence of a zero vector : $v + 0 = v$. (There is only one zero vector: $0_2 = 0_2 + 0_1 = 0_1 + 0_2 = 0_1$.)
- Associativity of addition : $v + (w + y) = (v + w) + y$.
- Existence of additive inverses : $v + (-v) = 0$. (Additive inverses are unique: $(-v)_2 = 0 + (-v)_2 = (v + (-v)_1) + (-v)_2 = v + ((-v)_1 + (-v)_2) = v + ((-v)_2 + (-v)_1) = (v + (-v)_2) + (-v)_1 = 0 + (-v)_1 = (-v)_1$.)

- Identity multiplication :$1.v = v$.
- Associativity of scalar multiplication : $a.(b.v) = (ab).v$.
- Distributivity : $(a+b).v = a.v+b.v$ and $a.(v+w) = a.v+a.w$. (So $0.v = (0+0).v = 0.v + 0.v$. By additive inverses, $0.v = 0$. Moreover,$0 = (1-1).v = 1.v + (-1).v = v+(-1).v$. Hence $-v = (-1).v$. Also,$a.0 = 0$ (why ?).) Many others can be proved similarly.

Examples and non-examples

- $\mathbb{R}^n, \mathbb{C}^n$.
- The set of continuous functions from $[0,1]$ to $\mathbb{R}$ (or $\mathbb{C}$) under the usual addition and scalar multiplication operations.
- Polynomials of degree $\leq n$ with $\mathbb{F}$-coefficients. (Polynomials of degree *exactly* n do NOT form a vector space.)
- Polynomials with integer coefficients do NOT form a vector space.
- $m \times n$ matrices with complex/real entries.
- The set of all differentiable functions $x, y : \mathbb{R} \to \mathbb{R}$ satisfying $x' = 2x + 3y, y' = 4x + 5y$ form a vector space over $\mathbb{R}$.

4 Subspaces

Subspaces are subsets that form vector spaces in their own right with the same operations. One can prove that is enough for just *closure* to hold to be a subspace. For instance, the set of all diff functions satisfying the ODE above forms a subspace of the set of all differentiable functions.On the other hand, the set of non-zero reals is NOT a subspace of reals. Given a set S,the subspace *generated/spanned* by it is the space $L(S)$ (also called the linear span of S) consisting of*finite* linear combinations $\sum_{k=1}^{N} c_k s_k$ of elements of S.If $S = \phi$, $L(S) := \{0\}$.

A set $S \subset V$ is called linearly dependent ifthere is a finite subset $x_1, x_2, \ldots, x_k$ (distinct) and scalars $c_1, \ldots, c_k$ not all zero such that$\sum_k c_k x_k = 0$.
It is independent if it is not dependent, i.e.,whenever $\sum_k c_k x_k = 0$, all $c_k = 0$.
For instance, if $0 \in S$, it is dependent and the empty set is independent. The set t^n is independent.So is e^{inx} (We shall see an alternate proof using more machinery later). If $S = \{x_1, \ldots, x_k\} \subset V$ is independent,then *any* set of $k+1$ vectors in $L(S)$ is dependent.

A set $S \subset V$ is called a basis if it is independent and spans V. It turns out (not easy !) that *every* vector space V has a basis.
The catch is that it need not be a finite basis ! (For instance, the set of continuous functions has infinitely many linearly independent elements (like t^n for instance).So it cannot have a finite basis.)

Those vector spaces that admit a finite basis are called *finite-dimensional* vector spaces. We shall study only such spaces in this class.Infinite-dimensional ones are also useful in mathematics (and in physics, engineering, etc for that matter) but require more complicated tools.The study of certain infinite-dimensional vector spaces is called *Functional Analysis*.

5 Dimension

If V is finite-dimensional, then any finite basis has the same number of elements. This number is called the dimension of V. (Warning ! $\mathbb{C}^n$ can *also* be thought of as an $\mathbb{R}$-vector space but with a dimension of $2n$ instead of n!)$\{0\}$ has dimension 0. *Any* linearly independent set of $k < \dim(V)$ elements can be extended to a basis of V.Moreover, any set of $\dim(V)$ linearly independent elements forms a basis of V. Often, one considers an *ordered* basis, i.e., a basis written in a specified order.In that case, every vector $v = \sum_k c_k e_k$. The (uniquely determined) numbers c_k are called *components* of v relative to the ordered basis $\{e_1, \ldots, e_n\}$.