1 Recap

- Recalled the definition of a vector space V over a field $\mathbb{F}$ (usually reals or complex numbers). By the way, using induction and associativity, $\sum_{i=1}^{n} c_i v_i$ can be defined unambiguously.

- Subspaces and the linear span of a set.

- Finite-dimensional vector spaces, dimension, and the notion of a basis.

- Ordered bases and components.

2 Linear Transformations/Maps

Recall that one of the aims of defining vector spaces was to solve linear equations in general. Often, this involves adding/subtracting to get "new variables" that can be solved for. Moreover, given two vector spaces, are they the same vector space in disguise ? (Is Voldemort being called Tom Riddle ?) To this end, we neeed to define maps/functions/transformations between vector spaces that preserve the vector space structure. Recall that if V, W are vector spaces (over the same field), then a function $T : V \to W$ is called a linear transformation/linear map if $T(av) = aT(v)$ for all $a \in \mathbb{F}, v \in V$ and $T(v + w) = T(v) + T(w)$ or alternatively, $T(av + bw) = aT(v) + bT(w)$. So $T(0) = T(0.v) = 0.T(v) = 0$.
The image $T(V)$ is a *subspace* of W : If $T(v), T(w) \in T(V)$, then $aT(v) + bT(w) = T(av + bw) \in T(V)$. $T(\sum_i c_i v_i) = \sum_i T(c_i v_i)$: We prove by induction. For $n = 1$, it follows from definition. Assume truth for n. For $n + 1$, $T(\sum_{i=1}^{n+1} c_i v_i) = T(\sum_{i=1}^{n} c_i v_i + c_{n+1} v_{n+1}) = \sum_{i=1}^{n} c_i T(v_i) + c_{n+1} T(v_{n+1})$.

Examples and Non-examples of Linear Transformations:

- If $V = \mathbb{R}^n, W = \mathbb{R}^m$, and A is an $m \times n$ real matrix, then $T(x) = Ax$, i.e, $(T(x))_i = \sum_j A_{ij} x_j$ is a linear map. (In fact, *all* linear maps between these particular V and W arise this way.) However, $T(x) = Ax + b$ is NOT linear.

- If V is the space of continuous $f : [0, 1] \to \mathbb{R}$ and $W = \mathbb{R}$, then $T(f) = \int_0^1 f(x)dx$ is a linear map. However, $\int_0^1 f^2 dx$ is NOT linear.

- If V is the space of real polynomials of degree ≤ 5 and $W = \mathbb{R}$, then $T(p) = p(0)$ is a linear map. (It is called an evaluation map.)

3 Algebraic operations on Linear maps

If V, W are vector spaces over the same field, $T, H : V \to W$ are linear maps, then $T + H$ is linear and so is cT for all $c \in \mathbb{F}$. One can verify that the set of all linear maps $L(V, W)$ forms a vector space in its own right. If V, W, X are vector spaces, and $T : V \to W$, $U : W \to X$ are linear maps, then $U \circ T : V \to X$ is linear. $R(ST) = (RS)T$, i.e., associativity holds. Moreover, $(R + S)T = RT + ST$ and $R(S + T) = RS + RT$.

4 Linear maps and matrices for finite-dimensional vector spaces

Given ordered bases $e_1, \ldots, e_n$ for V and $f_1, \ldots, f_m$ for W, $T(\sum_k c_k e_k) = \sum_k c_k T(e_k)$ and hence it is enough to know what $T(e_k)$ are. Let $T(e_k) = \sum_j T_{jk} f_j$. Then $T(\sum_k c_k e_k) = \sum_k c_k \sum_j T_{jk} f_j = \sum_j (\sum_k T_{jk} c_k) f_j$.
The matrix T_{jk} determines T and vice-versa. The components c_k, if represented by a column vector (as is usually the case), go to a *new* component-column-vector d_j as $d = [T]c$. So to link linear maps and matrices, one needs to *choose* ordered bases for *both*, the image AND the target. Different choices of ordered bases give rise to *different* matrices representing the *same* linear map.

Examples of matrix-representation of linear maps

- The algorithm is as follows : The first column of the matrix is obtained by calculating $T(e_1)$ and writing its components as a column vector in the given ordered basis of the target. Likewise for the other columns.
- So if we consider the *differentiation* linear map from degree$\leq$ 2 polynomials to itself with an ordered basis $\{1, x, x^2\}$, $T(1) = 0 = 0.1 + 0.x + 0.x^2$, $T(x) = 1 = 1.1 + 0.x + 0.x^2$, and $T(x^2) = 2x = 0.1 + 2.x + 0.x^2$.
- Thus the matrix is $[T] = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$

Matrix operations:
If A, B are two $m \times n$ matrices with entries in $\mathbb{F}$, then $[A+B]_{ij} := [A]_{ij} + [B]_{ij}$ and if $c \in \mathbb{F}$, then $[cA]_{ij} := c[A]_{ij}$.
If we choose ordered bases $\{e_i\}, \{f_j\}, \{g_k\}$ for f.d vector spaces V, W, X, and if $T : V \to W$, $U : W \to X$ are linear maps, then then we get two matrices $[T]$ and $[U]$ representing the maps. It turns out that $U \circ T$ is represented by $[U][T]$ where multiplication is in the sense of *matrix multiplication*, i.e., $([A][B])_{ij} = \sum_k [A]_{ik} [B]_{kj}$: Indeed, $U(T(e_i)) = U(\sum_j T_{ji} f_j) = \sum_j T_{ji} U(f_j) = \sum_j T_{ji} \sum_k U_{kj} g_k = \sum_{j,k} U_{kj} T_{ji} g_k$. In fact, matrix multiplication is defined so that this happens.

Properties of matrix multiplication:

- $A(BC) = (AB)C$ whenever it makes sense.
- $(A+B)C = AC + BC$ and $C(A+B) = CA + CB$ whenever it makes sense.
- The simplest proof is to interpret each of the matrices as linear maps between appropriate vector spaces and use the fact that $[U \circ T] = [U][T]$.

Null space/Kernel of a linear map: Suppose we consider the equation $2x + 3y = 1$. How many solutions does it have ? Infinitely many. What about $2x + 3y = 1, 4x + 6y = 3$? Zero. What about $2x + 3y = 1, 4x + 6y = 2$? Infinitely many. What about $2x + 3y = 1, x - y = 0$? Exactly one. More generally, $T : V \to W$ need not be surjective or injective. Suppose $T(v) = w$. How many solutions does this equation have if it has one ? Note that if $T(v_1) = T(v_2) = w$, then $T(v_1 - v_2) = 0$. Motivated by this observation, we define the *null space* $N(T) \subset V$ as the set $v \in V$ so that $T(v) = 0$. If $T(v) = T(w) = 0$, then $T(av + bw) = aT(v) + bT(w) = 0$ and hence $N(T)$ is a *subspace* of V.