1 Recap

- Linear maps and their connection with matrices.
- Nullity.

2 Nullity and Rank

Note that T is $1-1$ if and only if $N(T) = \{0\}$: If T is $1-1$, then $T(0) = 0$ and hence $N(T) = \{0\}$. If $N(T) = \{0\}$ and if $T(v) = T(w)$, then $T(v-w) = 0$ and hence $v - w \in N(T)$ implying that $v = w$. Hence T is $1-1$.
The dimension of the null space/Kernel of $T : V \to W$ (where V, W are f.d) is called the *nullity* of T. It measures the failure of injectivity of T. It is basically the number of "free parameters". The dimension of the image is called the *rank* of T. It measures the failure of surjectivity of T.
An important result is the Nullity-Rank Theorem : The Nullity of T + the Rank of T equals $\dim(V)$.
Proof: Suppose $e_1, \ldots, e_k$ is a basis of $N(T)$. Extend it to a basis $e_{k+1}, \ldots, e_n$ of V. We claim that $f_1 = T(e_{k+1}), f_2 = T(e_{k+2}), \ldots, f_{n-k} = T(e_n)$ is a basis of $R(T)$.
Indeed, f_i are linearly independent : If $\sum_i c_i f_i = 0$, then $T(\sum_{i=k+1}^n c_i e_i) = 0$ and hence $\sum_{i=k+1}^n c_i e_i \in N(T)$ which implies that $\sum_{i=k+1}^n c_i e_i = \sum_{i=1}^k d_i e_i$ and hence by linear independence, $c_i = d_j = 0 \ \forall \ i, j$.
Moreover, f_i span $R(T)$: $T(v) = T(\sum_{i=1}^n c_i e_i) = T(\sum_{i=1}^k c_i e_i) + T(\sum_{i=k+1}^n c_i e_i) = 0 + \sum_i c_i f_i$.

3 Inner products

How does one prove that $e^{ix}, e^{2ix}, \ldots, e^{inx}$ are linearly independent in the vector space of continuous functions on $[0, 2\pi]$? Induction is one way. A nicer way is (due to Fourier) : If $\sum_k c_k e^{ikx} = 0$, then multiply by e^{-imx} and integrate from 0 to 2π. Then $c_m 2\pi = 0$ and hence $c_m = 0$. This proof is like taking a dot product with a bunch of vectors and isolating each component. So it is fruitful to define the notion of a dot product on arbitrary vector spaces (over $\mathbb{R}$ or $\mathbb{C}$. This notion does not make sense for all fields).

Let V be a vector space over $\mathbb{R}$. An inner product (a dot product) is a function $\langle , \rangle : V \times V \to \mathbb{R}$ that satisfies the following properties.

- Symmetry : $\langle x, y \rangle = \langle y, x \rangle$.
- Additive Linearity : $\langle x, y + z \rangle = \langle x, y \rangle + \langle x, z \rangle$.
- Scalar linearity : $\langle cx, y \rangle = c\langle x, y \rangle$.
- Positivity : $\langle x, x \rangle > 0$ when $x \neq 0$. (Note that $\langle 0, 0 \rangle = 0\langle 0, 0 \rangle = 0$.)

Let V be a vector space over $\mathbb{C}$. An inner product (a dot product) is a function $\langle , \rangle : V \times V \to \mathbb{C}$ that satisfies the following properties.

- Hermitian symmetry : $\langle x, y\rangle = \overline{\langle y, x\rangle}$.
- Additive Linearity : $\langle x, y+z\rangle = \langle x, y\rangle + \langle x, z\rangle$.
- Scalar sesquilinearity : $\langle cx, y\rangle = c\langle x, y\rangle$.
- Positivity : $\langle x, x\rangle > 0$ when $x \neq 0$. (Note that $\langle 0, 0\rangle = 0\langle 0, 0\rangle = 0$.)

Examples and non-examples:

- The "usual" dot product in $\mathbb{R}^n$. A nice way of writing it is :$\langle v, w\rangle = v^T w$, where v^T is the *transpose* of v, i.e., one converts rows to columns to get a new matrix.
- $\langle x, y\rangle = x_1\bar{y}_1 + x_2\bar{y}_2 + \ldots$ in $\mathbb{C}^n$ is an inner product (corresponding to $\langle x, y\rangle = x^T\bar{y}$) but $x_1y_1 + x_2y_2 + \ldots$ is NOT.
- On $\mathbb{R}^2$: $\langle v, w\rangle = 2v_1w_1 + v_1w_2 + w_1v_2 + v_2w_2$ is an inner product but $v_1w_1 + \frac{1}{2}(v_1w_2 + v_2w_1) + \frac{1}{8}v_2w_2$ is NOT.
- On the space of continuous real-valued functions on $[0, 1]$: $\langle f, g\rangle = \int_0^1 f(t)g(t)dt$ is an inner product.
- More generally, given a positive continuous function $w(t)$, $\int_0^1 w(t)f(t)g(t)dt$ is an inner product.
- On the space of continuous complex-valued functions on $[0, 1]$: $\langle f, g\rangle = \int_0^1 f(t)\bar{g}(t)dt$ is an inner product.