1 Recap

- Nullity-Rank.
- Inner products and examples/non-examples.

2 Inner products and the Cauchy-Schwarz inequality

Suppose V is a real f.d vector space, and $\langle , \rangle$ is an inner product. Let $e_1, \ldots, e_n$ be a basis. Then $\langle v, w\rangle = \langle \sum_i v_i e_i, \sum_j w_j e_j\rangle = \sum v_i w_j \langle e_i, e_j\rangle$. Define the matrix $H_{ij} = \langle e_i, e_j\rangle$. Then $H_{ij} = H_{ji}$, i.e., $H = H^T$. Such a square matrix is called *symmetric.* Thus, $\langle v, w\rangle = v^T H w$. Since $\langle v, v\rangle > 0$ when $v \neq 0$, $v^T H v \geq 0$ with equality if and only if $v = 0$. Such a matrix H is called *positive-definite.* It turns out that *every* inner product on V is obtained through positive-definite matrices this way (HW).

The Cauchy-Schwarz inequality:
In $\mathbb{R}^2$ and $\mathbb{R}^3$, one can *prove* using elementary geometry/calculus that $(v.w)^2 = (v.v)(w.w)\cos^2(\theta)$. As a consequence, $(v.w)^2 \leq (v.v)(w.w)$ with equality if and only if $\theta = 0$, that is, v and w are parallel, i.e., $v = \lambda w$ or $w = \lambda v$. While there is no elementary geometric picture for general vector spaces, one can still prove this inequality (the Cauchy-Schwarz inequality) : Suppose $\langle , \rangle$ is an inner product on a real or a complex vector space V (not necessarily f.d) then $|\langle v, w\rangle|^2 \leq \langle v, v\rangle\langle w, w\rangle$ with equality if and only if $v = \lambda w$ or $w = \lambda v$ for some $\lambda \in \mathbb{F}$. As a consequence, $(\int_0^1 fg dx)^2 \leq \int_0^1 f^2 dx \int_0^1 g^2 dx$!

A small interlude before the proof of CS: Define a "norm" $\|x\| = \langle x, x\rangle^{1/2}$. So CS is $|\langle x, y\rangle| \leq \|x\|\|y\|$. The norm obeys the following:

- Positivity: $\|x\| \geq 0$ with equality if and only if $x = 0$. (Easy)
- Homogeneity: $\|cx\| = |c|\|x\|$ (Easy)
- Triangle Inequality (TI) : $\|x + y\| \leq \|x\| + \|y\|$. (In fact, equality holds in the TI iff x, y are parallel.) : $\|x + y\|^2 = \langle x + y, x + y\rangle = \|x\|^2 + \|y\|^2 + \langle x, y\rangle + \langle y, x\rangle$. By CS and completing the square, we get the result.

In fact, one can define a norm on a vector space without even defining an inner product but not all norms arise out of an inner product. (An example is the "taxi-cab" norm).

Proof of the Cauchy-Schwarz inequality in the real case: Assume that $w \neq 0$ w LOG. (If $w = 0$ then $\langle v, w\rangle^2 = 0 = \langle v, v\rangle\langle w, w\rangle$ and $w = 0v$.) This technique is called "Arbitrage" (Terence Tao's term) : The only inequality available to us is $\langle x, x\rangle \geq 0$. The only way to get v, w into the picture is to take a linear combination. So for every positive real $t > 0$, we have a wimpy little inequality for free : $\langle v + tw, v + tw\rangle \geq 0$. We shall choose the "worst-case" t and apply the silly inequality above, i.e., we shall minimise $f(t) = \langle v + tw, v + tw\rangle^2$. $f(t) = \|v\|^2 + t^2\|w\|^2 + 2t\langle v, w\rangle$. $f'(t) = 0$ implies that $t = -\frac{\langle v,w\rangle}{\|w\|^2}$. So $\|v\|^2 + \frac{\langle v,w\rangle^2}{\|w\|^2} - 2\frac{\langle v,w\rangle^2}{\|w\|^2} \geq 0$. Hence $\langle v, w\rangle^2 \leq \|v\|^2\|w\|^2$. Equality holds precisely when $v + tw = 0$, i.e., $v = -tw$. For the complex case, choose $t = -\frac{\langle v,w\rangle}{\|w\|^2}$ as before.