1 Recap

- Limit and continuity laws.
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2 Limits and continuity

Composition:
Theorem: Let $\vec{g} : U \subset \mathbb{R}^m \to \mathbb{R}^n$ be a function that is continuous at $\vec{a} \in U$. Let $\vec{g}(U) \subset V \subset \mathbb{R}^n$ and let $\vec{f} : V \subset \mathbb{R}^n \to \mathbb{R}^p$ be a function that is continuous at $\vec{g}(\vec{a}) \in V$. Then $\vec{f} \circ \vec{g} : U \to \mathbb{R}^p$ is continuous at $\vec{a}$.
Proof: Note that $\|\vec{f}(\vec{y}) - \vec{f}(\vec{g}(\vec{a}))\| < \epsilon$ whenever $\|\vec{y} - \vec{g}(\vec{a})\| < \tilde{\delta}$ and $\vec{y} \in V$. Choose δ to be so small that $\|\vec{g}(\vec{x}) - \vec{g}(\vec{a})\| < \tilde{\delta}$ whenever $\|\vec{x} - \vec{a}\| < \delta$. Thus we are done. □Thus, $\sin(x^2y)$, $\ln \frac{x-y}{x+y}$, $\ln \cos^2(x^2 + y^2)$ etc, are continuous wherever they are defined.

3 Derivatives of scalar fields w.r.t vectors

Suppose we consider a scalar field like the temperature $T(x, y, z)$ of a room. Unlike 1-variable calculus, we can ask "How fast does T change when we move a little in a certain direction?" The answer can of course depend on the direction. To even make sense of this question, we must be allowed to move a little in *all* directions, i.e., the point under consideration must be an *interior* point of the domain.
Def: Given a scalar field $f : U \subset \mathbb{R}^n \to \mathbb{R}$, an interior point $\vec{a} \in U$, and a vector $\vec{v} \in \mathbb{R}^n$, f is said to differentiable along $\vec{v}$ if $\lim_{h\to 0} \frac{f(\vec{a}+h\vec{v})-f(\vec{a})}{h}$ exists. This number is denoted as $\nabla_{\vec{v}} f$. If $\|\vec{v}\| = 1$, this number is called the *directional* derivative along the direction $\vec{v}$.

Examples:

- If $\vec{v} = 0$, then $\nabla_{\vec{v}} f(\vec{a}) = 0$. Indeed, this follows easily from the definition.
- Suppose f is linear, then $\frac{f(\vec{a}+h\vec{v})-f(\vec{a})}{h} = \vec{f}(\vec{v})$ for all h and hence f is differentiable along all vectors with $\nabla_{\vec{v}} f(\vec{a}) = f(\vec{v})$.
- The above derivative can also be thought of as a one-variable derivative. Indeed, let $g(t) = f(\vec{a} + t\vec{v})$. Then $\frac{f(\vec{a}+h\vec{v})-f(\vec{a})}{h} = \frac{g(h)-g(0)}{h}$. Thus g is differentiable at 0 if and only if $\nabla_{\vec{v}} f(\vec{a})$ exists and equals $g'(0)$.
- Moreover, $\nabla_{s\vec{v}} f(\vec{a})$ exists whenever $\nabla_{\vec{v}} f(\vec{a})$ does and equals $s\nabla_{\vec{v}} f(\vec{a})$: Indeed, let $h(t) = f(\vec{a} + t\vec{v})$ and $g(t) = h(st) = f(\vec{a} + ts\vec{v})$. Then by the chain rule $g'(0)$ exists and equals $sh'(0) = s\nabla_{\vec{v}} f(\vec{a})$.

Assume that $\nabla_{\vec{v}} f(\vec{a} + t\vec{v})$ exists for all $0 \le t \le 1$.
Theorem: Then there exists a real number $\theta \in (0, 1)$ such that $f(\vec{a} + \vec{v}) - f(\vec{a}) = \nabla_{\vec{v}} f(a + \theta\vec{v})$.
Proof: Let $g(t) = f(\vec{a}+t\vec{v})$. Then applying the 1-dim MVT to g, we see that $g(1)-g(0) = g'(\theta)$. Thus we are done. □

Def: When $\vec{v} = e_i$, $\nabla_{\vec{v}} f(\vec{a})$ if it exists is called the *partial derivative* $\frac{\partial f}{\partial x_i}(\vec{a})$ or $f_i(\vec{a})$ w.r.t x_i and f is said to be partially differentiable w.r.t x_i. Spoiler alert: All known laws of nature are Partial Differential Equations (PDE) for something or the other (not necessarily for scalar fields). In effect, a partial derivative is simply the usual derivative w.r.t one variable whilst keeping the others fixed (Why?).
Examples:

1. A polynomial has directional derivatives in all directions at all points because $g(t)$ is a polynomial and hence differentiable.

2. More generally, if $f_i(x_i)$ are differentiable, then a sum of terms of the form $f_1(x_1)f_2(x_2)\ldots$ has directional derivatives in all directions.

3. One can easily compute partials in specific examples: For instance, if $T(P,V) = \frac{PV}{R}$, where R is a constant, then T is partially differentiable w.r.t P, V. Its partials are $\frac{\partial T}{\partial V} = \frac{P}{R}$, $\frac{\partial T}{\partial P} = \frac{V}{R}$.

4. A rational function is partially diferentiable w.r.t all x_i wherever it makes sense. In fact, it has directional derivatives also in all directions wherever it makes sense.

If the partials w.r.t all coordinates exist then we get a vector-valued function called the *gradient* of f: $\nabla f(x) = (f_{x_1}, f_{x_2}, \ldots)$. One can further ask if each component of the gradient has partial derivatives. In this manner, we get higher partial derivatives like $f_{xy} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x} f_y$ and $f_{yx} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y} f_x$ (which may in fact be different from each other!). In the above example, $\frac{\partial^2 T}{\partial P^2} = 0 = \frac{\partial^2 T}{\partial V^2}$, and $\frac{\partial^2 T}{\partial P \partial V} = \frac{\partial^2 T}{\partial V \partial P} = \frac{1}{R}$.

In one-variable calculus, differentiability at a point implied continuity at that point. The way it works is as follows: $|f(x+h) - f(x)| = |\frac{f(x+h)-f(x)}{h}||h|$. Now the definition of differentiability implies that there exists a δ so that whenever $0 < |h| < \delta$, $|\frac{f(x+h)-f(x)}{h}| < |f'(x)| + 1$. Choose δ to be possibly smaller so that $|\frac{f(x+h)-f(x)}{h}||h| < \epsilon$. We are done.
Naively, we would expect that if f is differentiable w.r.t all $\vec{v}$ at $\vec{a}$, then since the limit $\frac{f(\vec{a}+h\vec{v})-f(\vec{a})}{h}$ exists as $h \to 0$, we might be able to conclude that the numerator goes to 0 and hence f is continuous.
The problem is with the "hence" part. It isn't true that just because the limit exists along all lines it exists in the multivariable sense.

Example: Let $f(x,y) = \frac{xy^2}{x^2+y^4}$ if $x \neq 0$ and $f(0,y) = 0$. By what we discussed, f is certainly continuous when $x \neq 0$. Moreover, all of its directional derivatives exist when $x \neq 0$. At $x = 0$, $f(0,y) = 0$ and hence all its directional derivatives do exist. When $x = 0$ but $y = b \neq 0$, $|f(x,y) - f(0,b)| = 0$ if $x = 0$ and if $x \neq 0$, it is less than $\frac{|x|}{y^2} < \frac{4|x|}{b^2}$ if $|y| > \frac{|b|}{2}$. Thus if $\delta < \frac{|b|}{2}$, such is the case. Moreover, if $\delta < \frac{\epsilon b^2}{4}$, then $|f(x,y) - f(0,b)| < \epsilon$. Thus f is continuous away from the origin.
At the origin unfortunately $f(x,y)$ is NOT continuous:
Indeed, along $x_n = y_n^2 = \frac{1}{n}$ $f(x_n, y_n) = \frac{1}{2}$, whereas $f(0,0) = 0$ by definition. So it is NOT good enough for all directional derivatives to exist.