1 Recap

- C^1 implies differentiability.
- Statement of the chain rule (for $f(\vec{r}(t))$).

2 Chain rule

Examples:

- Suppose $f(x,y) = x^2 - y^2$ and $\vec{r}_1(t) = (t, 2t)$ and $\vec{r}_2(t) = (2t, 4t)$. Then $\nabla f = (2x, -2y)$ and $\vec{r}_1' = (1,2)$, $\vec{r}_2' = (2,4)$. Thus $\frac{df(x_1(t),y_2(t))}{dt} = -6t$ and $\frac{df(x_1(t),y_2(t))}{dt} = -24t$. In other words, the choice of parameterisation can *affect* the result. (Warning: At a point say $(1,2)$, for $\vec{r}_1$, $t=1$ and for $\vec{r}_2$, $t = \frac{1}{2}$. Thus the derivatives of f are $-6, -12$ respectively. On the other hand, $\frac{df}{ds}$ is the same for both paths.)

- Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a differentiable scalar field such that $\nabla f \neq \vec{0}$ anywhere. Let $c \in \mathbb{R}$ a constant such that $f(x,y) = c$ describes a differentiable curve C having a well-defined unit tangent vector at each point. Prove that the following hold.

  1. The gradient vector ∇f is *normal* to C.
  2. The directional derivative $\frac{df}{ds}$ along C is 0.
  3. The directional derivative of f at any point on C is highest in the normal direction to C.

  Let $\vec{v}$ be the unit tangent vector at a point (x_0, y_0) on C, i.e., $f(x_0,y_0) = c$. By the assumption of a well-defined unit tangent , the curve C can be parameterised as $\vec{r}(t) = (x(t), y(t))$ where the tangent vector at $(x_0,y_0) = (x(0), y(0))$ is $\vec{v}$, i.e., $(x'(t), y'(t)) = \vec{v}$. Thus $\frac{df}{ds}$ at $t = 0$, i.e., at (x_0, y_0) is $\frac{df(x(t),y(t))}{dt} = 0$ because $f(x(t), y(t)) = c$ for all t. This proves the second part. Moreover, $\frac{df(x(t),y(t))}{dt} = \langle \nabla f, \vec{v} \rangle = 0$ and hence $\nabla f(x_0, y_0)$ is perpendicular to the tangent, i.e., it is normal to C. Lastly, by Cauchy-Schwarz, the directional derivative at (x_0, y_0) is highest along $\nabla f(x_0, y_0)$ which we just proved is normal to C.

Proof: Recall that roughly, $f(\vec{r}(t+h)) - f(\vec{r}) \approx f(\vec{r} + h\vec{r}') - f(\vec{r}) \approx h\nabla_{\vec{r}'} f$. More rigorously, letting $g(t) = f(\vec{r}(t))$, $g(t_0 + h) - g(t_0) = f(\vec{a} + \vec{y}) - f(\vec{a})$ where $\vec{a} = \vec{r}(t_0)$, $\vec{y} = \vec{r}(t_0+h) - \vec{r}(t_0)$. Using the definition of differentiability of f, $f(\vec{a}+\vec{y}) = f(\vec{a}) + \langle \nabla f(\vec{a}), \vec{y} \rangle + \|\vec{y}\| E(\vec{a}, \vec{y})$, where $E \to 0$ as $\|\vec{y}\| \to 0$. So $\frac{g(t_0+h)-g(t_0)}{h} = \frac{\langle \nabla f(\vec{r}(t_0)), \vec{r}(t_0+h)-\vec{r}(t_0) \rangle}{h} + \frac{\|\vec{r}(t_0+h)-\vec{r}(t_0)\|}{h} E(\vec{a}, \vec{r}(t_0 + h) - \vec{r}(t_0))$. This goes to the correct answer as $h \to 0$. Indeed $|\frac{\|\vec{r}(t_0+h)-\vec{r}(t_0)\|}{h}| E \to 0$.

Level sets and tangent planes: Whenever $f : \mathbb{R}^n \to \mathbb{R}$ is a function, the set $\vec{r} \in \mathbb{R}^n$ such that $f(\vec{r}) = c$ is called a *level set* of f. (If $n = 2$, it is a called a level curve or a contour line. If $n = 3$, it is called a level surface.) These occur as equipotential surfaces in physics. Even if f is C^1, this level set need not always be a “nice smoothly varying” object. For instance, take $xy = 0$ in $\mathbb{R}^2$ or $x^2 + y^2 = z^2$ in $\mathbb{R}^3$. If f is C^1, and on the *entire* level set $f^{-1}(c)$, $\nabla f \neq \vec{0}$ (a *regular* level set), then it turns out (by a theorem called the implicit function theorem) that near any point on this level set the level set can be treated as a graph of a function. In particular, the tangent planes exist at any point. Near a point $\vec{a}$ if $\vec{r}(t)$ is a C^1 curve passing through $\vec{a}$, i.e., $\vec{r}(0) = \vec{a}$, and if $\vec{r}'(0) = \vec{v}$, then since $f(\vec{r}(t)) = c$ for all t, its derivative is zero and hence $\langle \nabla f(\vec{a}), \vec{v} \rangle = 0$. This means that $\nabla f(\vec{a})$ is perpendicular to every tangential vector $\vec{v}$. If the level set is regular, then since tanget planes exist, their equation is $(\vec{r} - \vec{a}).\nabla f(\vec{a}) = 0$. For example, if $g(x, y, z) = z - f(x, y)$, then the level set $g = 0$ corresponds to the graph of $f(x, y)$. The tangent plane can be easily calculated to be $z = f(a, b) + \frac{\partial f}{\partial x}(a, b)(x - a) + \frac{\partial f}{\partial y}(a, b)(y - b)$ which is precisely the linear approximation of f.