# 1 Recap

- Second-derivative test in multivariable calculus.

# 2 Line integrals

Suppose a force $\vec{F}$ field (like an electric field) is acting in a region. What is the work done by this force to move a particle along a path $\vec{r}(t)$ from $\vec{r}(0)$ to $\vec{r}(1)$? Naively, in time $dt$, the particle moves by $\vec{dr} = \vec{r}'(t)dt$ and hence the work is $dW = \langle \vec{F}, \vec{dr} \rangle = \langle \vec{F}(\vec{r}(t)), \vec{r}'(t) \rangle dt$. The corresponding integral is the total work done. We want to put all of this on a rigorous footing.
Def: A continuous path in $\mathbb{R}^n$ is a continuous function $\vec{\alpha}(t) : [a, b] \to \mathbb{R}^n$. A $C^1$ path is one where $\vec{\alpha}(t)$ is $C^1$. A piecewise $C^1$ path is one for which $[a, b]$ can be partitioned into finitely many sub-intervals such that $\vec{\alpha}(t)$ is $C^1$ on each of them.
Def: Let $\vec{\alpha}(t)$ be a piecewise $C^1$ path on $J = [a, b]$ in $\mathbb{R}^n$. Let $\vec{F}$ be a vector field defined on the image of $\vec{\alpha}$ and is bounded. The line integral of $\vec{F}$ along $\vec{\alpha}$ is defined as $\int \langle \vec{F}, d\vec{\alpha} \rangle = \int_a^b \langle \vec{F}(\vec{\alpha}(t)), \frac{d\vec{\alpha}}{dt} \rangle dt$ whenever the integral exists. We can also define the line integral if $\vec{F}$ is a function that depends on $\vec{r}$ and $t$ in the same way. (For instance, if $\vec{F} = m\vec{\alpha}''$.)
In $\mathbb{R}^3$ it is also denoted as $\int_{\vec{\alpha}} (F_1 dx + F_2 dy + F_3 dz)$.
Example: Let $\vec{F} = \sqrt{y}\hat{i} + (x^3 + y)\hat{j}$ for all $(x, y)$ with $y \geq 0$. Calculate the line integral of $\vec{F}$ from $(0, 0)$ to $(1, 1)$ along each of the two paths: $\vec{\alpha}_1(t) = (t, t)$, $\vec{\alpha}_2(t) = (t^2, t^3)$ where $0 \leq t \leq 1$. Firstly, $\vec{F}$ is continuous on its domain and $\vec{\alpha}_i$ are $C^1$. Hence the integral exists. $d\vec{\alpha}_1(t) = (1, 1)$, $d\vec{\alpha}_2(t) = (2t, 3t^2)$. Moreover, $\vec{F}(\vec{\alpha}_1(t)) = (\sqrt{t}, t^3 + t)$ and $\vec{F}(\vec{\alpha}_2(t)) = (t^{3/2}, t^6 + t^3)$. Thus the integrals are $\int_0^1 (\sqrt{t} + t^3 + t)dt = \frac{17}{12}$ and $\int_0^1 (2t^{5/2} + (t^6 + t^3)3t^2)dt = \frac{59}{42}$. Thus the line integral can depend on the path taken.

What if we choose $\vec{\beta}(t) = (t^2, t^2)$? Then $d\vec{\beta} = (2t, 2t)$ and hence the integral is $\int_0^1 (t, t^6 + t^2).(2t, 2t)dt = \int_0^1 2t^2 + 2t^7 + 2t^3 dt = \frac{17}{12}$ which is precisely the integral over $\vec{\alpha}_1$! This suggests that the line integral *may* be invariant under reparametrisation.
The line integral satisfies linearity: $\int (a\vec{F} + b\vec{G}).d\vec{\alpha} = a \int \vec{F}.d\vec{\alpha} + b \int \vec{G}.d\vec{\alpha}$ and additivity: $\int_{\vec{\alpha}} \vec{F}.d\vec{r} = \int_{\vec{\alpha}_1} \vec{F}.d\vec{r} + \int_{\vec{\alpha}_2} \vec{F}.d\vec{r}$ if $\vec{\alpha}(t) = \vec{\alpha}_1$ for $t \in [a, c]$ and $\vec{\alpha}(t) = \vec{\alpha}_2$ for $t \in [c, b]$. The proofs are easy.
Let $u(t) : [a, b] \to [c, d]$ be a $C^1$ function such that $u'(t) \neq 0$ for all $t \in [a, b]$. $u$ is $1 - 1$ because either $u'(t) > 0$ for all $t$ or $u'(t) < 0$ for all $t$. So $t$ is a function of $u$ and it turns out that $t$ is $C^1$ in $u$. Such a $u$ is called a change of parameter. If $u' > 0$ for all $t$, $u$ is said to *preserve orientation* and *reverse orientation* if $u' < 0$ for all $t$.
The paths $\vec{\alpha}(u) : [c, d] \to \mathbb{R}^n$ and $\vec{\beta}(t) : [a, b] \to \mathbb{R}^n$ related by $\vec{\beta}(t) = \vec{\alpha}(u(t))$ are said to be reparametrisations of each other. Moreover, their ranges/images are the same geometric object in $\mathbb{R}^n$. That is, they are two paths parametrising the same *curve* $C$. If $u$ is orientation-preserving, then $\vec{\alpha}, \vec{\beta}$ are said to trace out the curve $C$ in the same direction as opposed to the opposite direction for orientation-reversing $u$.
Theorem: Let $\vec{\alpha}, \vec{\beta}$ be piecewise $C^1$ paths that are reparametrisations of each other. Then $\int_C \vec{F}.d\vec{\alpha} = \int_C \vec{F}.d\vec{\beta}$ if they trace out $C$ in the same direction and $\int_C \vec{F}.d\vec{\alpha} = -\int_C \vec{F}.d\vec{\beta}$ if

they do so in the opposite direction.
Proof: It is enough to prove it for $C^1$ paths by additivity. If $u' > 0$, then $u(a) = c, u(b) = d$. Thus, by substitution in the integral $\int_c^d \vec{F}(\vec{\alpha}(u)).d\vec{\alpha}(u)du$ we get $\int_a^b \vec{F}(\vec{\beta}(t)).d\vec{\alpha}(u(t))u'(t)dt$. By the chain rule, $d\vec{\beta}(t) = d\vec{\alpha}(u(t))u'(t)$. Thus we are done. If $u' < 0$, the sign changes because $u(a) = d$, $u(b) = c$.
In the example above, the Work done seemed to depend on the curve connecting the points. Its sign actually also depends on the parametrisation used for the curve. Forces for which the work is independent of the path taken ( as long as they trace out the same direction) are called *conservative* forces. The example above is not conservative.
Paths for which $\vec{\alpha}(a) = \vec{\alpha}(b)$ are called *closed*. Paths that are $1-1$ are called *simple*.
Work-Energy Theorem: The work done is equal to the change in the Kinetic energy. Proof: $\vec{F} = m\vec{r}''$. Thus, $\vec{F}.d\vec{r} = \frac{1}{2}m\frac{d}{dt}\|\vec{v}\|^2$. Integrating on both sides, we get the result.