1 Recap

- After linear algebra, we discussed ODE, in particular, $y' = 0, y' = ky$ and wrote a nonlinear competitive bacteria system. We "linearised" it for ease of analysis.

2 An ODE to a Grecian urn

Solve $x' = ax + by, y' = cx + dy$ on $\mathbb{R}$. The idea is to linearly change to u, v such that $u' = k_1 u, v' = k_2 v$. This idea is best implemented using matrices.
Let $v = \begin{bmatrix} x \\ y \end{bmatrix}$ and $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then we want to solve $\frac{dv}{dt} = Av$. Naively, one can attempt $v = e^{At}v_0$. Surprisingly, such an attempt works but we shall choose a different route.
Assume that A is diagonalisable over $\mathbb{R}$, i.e., $P^{-1}AP = D$ or $A = PDP^{-1}$. Then $\frac{dv}{dt} = PDP^{-1}v$. Let $w = P^{-1}v$. Then $\frac{dw}{dt} = Dw$. Thus $w_i'(t) = \lambda_i w_i$ and hence $w_i = w_i(0)e^{\lambda_i t}$ and $v = Pw$.
In other words, if A is diagonalisable, the set of solutions forms a vector space of dimension n spanned by $e^{\lambda_i t}Pe_i$. If $A = A^T$ then by the Spectral Theorem A is diagonalisable by an orthogonal matrix. In that case, the above strategy applies.

Example: Solve for all differentiable functions $x(t), y(t)$ on $\mathbb{R}$: $x' = 3x - 2y, y' = x$.
The matrix A is $A = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}$. It is not symmetric. We shall still attempt to diagonalise it. The eigenvalues are $1, 2$. Corresponding eigenvectors are $(1, 1)$ and $(2, 1)$ respectively. $P^{-1}AP = D$ where $P = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$. Let $w = P^{-1}v$. Then $\frac{dw}{dt} = Dw$. Hence $w(t) = ae^t e_1 + be^{2t}e_2$. Now $\begin{bmatrix} x \\ y \end{bmatrix} = Pw = ae^t \begin{bmatrix} 1 \\ 1 \end{bmatrix} + be^{2t} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

What if the eigenvalues are not real?
In this case, we need to solve $\frac{dw_i}{dt} = \lambda_i w_i$ where λ_i is complex. In other words, $\frac{dw}{dt} = (a + b\sqrt{-1})w$ where $w = u + \sqrt{-1}v$. One way to do it is $w(t) = w_0 e^{at}(\cos(bt) + \sqrt{-1}\sin(bt))$ and $w_0 \in \mathbb{C}$. While one can verify that this solves the equation how does one prove that it is *the* solution?
Ans: Same as before. $z(t) = w(t)e^{-at}(\cos(bt) - \sqrt{-1}\sin(bt))$. In your HW, you will show that the product rule still holds in this complex setting. Thus $z'(t) = 0$. Hence the real and imaginary parts of $z(t)$ are constants. That is, $v(t) = Pw$ where w and P are allowed to be complex.

3 Multivariable calculus

So far, we looked at either functions $f : U \subset \mathbb{R} \to \mathbb{R}$ and studied their continuity, differentiability, integrability, etc or *linear* functions $f : \mathbb{R}^n \to \mathbb{R}^m$ given by $f(x) = Ax$ and generalisations $T : V \to W$ to linear maps.

Most of life is nonlinear and multivariate (like the airflow around an aeroplane, some aspects of the stock market, the curvature of the earth, protein folding, Ricci flow for the Poincaré conjecture, etc). So we need to study the calculus of functions $f : U \subset \mathbb{R}^m \to \mathbb{R}^n$. When $n = 1$, the function is called scalar-valued or a scalar field and when $n > 1$ it is called vector-valued or a vector field.

Recall that a function $f : U \subset \mathbb{R} \to \mathbb{R}$ is continuous at $a \in U$ if for every $\epsilon > 0$ there exists a $\delta_\epsilon > 0$ such that whenever $|x - a| < \delta$ and $x \in U$, $|f(x) - f(a)| < \epsilon$.
So to even define continuity of $f : U \subset \mathbb{R}^m \to \mathbb{R}^n$, we need to be able to measure distances. To this end, we use the usual norm coming from the usual dot product $\vec{v}.\vec{w} = \sum_i v_i w_i$ and $\|v - w\| = \sqrt{(v_1 - w_1)^2 + \ldots}$.
The definition of continuity in one-variable is easiest to visualise when $U = (a, b)$. Therefore, we need to generalise the notion of an open interval to higher-dimensions.
Open ball: An open ball $B(\vec{a}, r) \subset \mathbb{R}^m$ centred at $\vec{a}$ with radius $r > 0$ is the set of all $\vec{x} \in \mathbb{R}^m$ such that $\|\vec{x} - \vec{a}\| < r$.
What is an open set in $\mathbb{R}$? Ans: Whatever it is, it must be natural/easy to do calculus on it! In other words whenever $x \in U$ it is helpful if $x + h \in U$ for all "small h". Thus, an open set $U \subset \mathbb{R}$ is one where all points in U are "inside" U, i.e., there is an *open interval* around each point that is wholly contained in U.
Motivated by the above, let $S \subset \mathbb{R}^n$ and $\vec{a} \in S$. $\vec{a}$ is called an *interior point* of S if there exists $r > 0$ such that the open ball $B(a, r)$ is contained in S.
Examples: 0 is *not* an interior point of $[0, 1]$. On the other hand, $\frac{1}{2}$ is an interior point. The point $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ is not an interior point of the set $\|x\| \leq 1$. $(0, \frac{1}{2})$ is an interior point though. No point is an interior point of the set $S = \{(0, 0), (0, 1), (1, 2), (4, 6)\}$.

A set $U \subset \mathbb{R}^m$ is said to be an *open set* if all points in U are interior points, i.e., for every $x \in U$, there is an open ball $B(x, r_x) \subset U$.
Some more terminology: For a general set $S \subset \mathbb{R}^m$ the *interior* is the collection of all interior points. So the interior can potentially be empty or potentially all of S (in which case S is an open set) or anything in between. For instance, the interior of $[0, 1]$ is $(0, 1)$. An open set containing a point $\vec{a} \in \mathbb{R}^m$ is often called a *neighbourhood* of $\vec{a}$.

Examples/Non-examples of open sets:

- An open ball $B(\vec{a}, r)$ is an open set: If $\vec{x} \in B(\vec{a}, r)$, then $\|\vec{x} - \vec{a}\| < r$. Now whenever $\|\vec{y} - \vec{x}\| < \frac{r - \|\vec{x} - \vec{a}\|}{2}$, we see that $\||y - \vec{a}\| \leq \|\vec{y} - \vec{x}\| + \|\vec{x} - \vec{a}\|$ by the triangle inequality. So $\|\vec{y} - \vec{a}\| \leq \frac{r + \|\vec{x} - \vec{a}\|}{2} < r$.

- The set $0 < x < 1, 0 < y < 2$ is open set in $\mathbb{R}^2$: If (a, b) is in the set then $0 < a < 1, 0 < b < 2$. Thus whenever $\|(x, y) - (a, b)\| < \frac{\min(a, 1-a, b, 2-b)}{2}$, then $|x - a| < \frac{\min(a, 1-a)}{2}$ and hence $0 < x < 1$ (why?) and likewise for y. Note that this set is $(0, 1) \times (0, 2)$.

- More generally, (HW) if $U \subset \mathbb{R}^n$ is open and $V \subset \mathbb{R}^m$ is open then $U \times V \subset \mathbb{R}^n \times \mathbb{R}^m$ is open.

- The set $[0, 1) \times (0, 2)$ is not open because 0 is not an interior point (why?)

- The set $(0,1) \times (0,2) \cup \{(5,6)\}$ is not open because $(5,6)$ is not an interior point.