1 Recap

- Solved linear systems of ODE.
- Defined open balls, interior points, and open sets. Gave examples and non-examples.

2 Exterior points, boundary points, and closed sets

A point $\vec{a} \in \mathbb{R}^n$ is said to be *exterior* to S if it is interior to the complement of S, i.e., there is an open ball around $\vec{a}$ containing no points of S. For instance, $(1,2)$ is exterior to $B((0,0),1)$. A point $\vec{a}$ that is neither interior nor exterior to S is called a boundary point. The set of all boundary points is called the boundary of S and is denoted as ∂S. For instance, the same rectangle is the boundary of $[0,1) \times (0,2)$ and of $(0,1) \times (0,2)$. A *closed set* is one whose complement is open. For instance, $[0,1] \times [0,2]$ is closed (why?). A set can neither be open nor closed! It can be *both* too! (For instance, $\mathbb{R} \subset \mathbb{R}$ is both!). One can prove (HW) that a set $S \subset \mathbb{R}^n$ is closed iff $S = int(S) \cup \partial S$.

3 Limits and continuity

Let $S \subset \mathbb{R}^n$ and $\vec{f} : S \to \mathbb{R}^m$ be a function. Let $\vec{a} \in \mathbb{R}^n$ and $\vec{b} \in \mathbb{R}^m$. We say that $\lim_{\vec{x} \to \vec{a}} \vec{f}(\vec{x}) = \vec{b}$ iff for every $\epsilon > 0$ there exists a $\delta > 0$ such that whenever $0 < \|\vec{x} - \vec{a}\| < \delta$ and $\vec{x} \in S$, $\|\vec{f}(\vec{x}) - \vec{b}\| < \epsilon$. Informally, $\lim_{\|\vec{x}-\vec{a}\| \to 0} \|\vec{f}(\vec{x}) - \vec{b}\| = 0$ in the one-variable sense. Just as in one-variable calculus f need not be defined at $\vec{a}$ for the limit to make sense. We can convert these definitions to $\vec{h} = \vec{x} - \vec{a}$ for convenience. A function is continuous at $\vec{x} = \vec{a} \in S$ if $\lim_{\vec{x} \to \vec{a}} \vec{f}(\vec{x}) = \vec{f}(\vec{a})$.
Proposition: The limit $\lim_{\vec{x} \to \vec{a}} \vec{f}(\vec{x}) = \vec{b}$ iff each of the limits $\lim_{\vec{x} \to \vec{a}} f_i(\vec{x})$ exists and equals b_i: It follows from $|f_j(\vec{x}) - b_j| \leq \|\vec{f}(\vec{x}) - \vec{b}\| \leq \sum_i |f_i(\vec{x}) - b_i|$ for every j (why?) $\square$
Sandwich law: If $\|\vec{f}(\vec{x})\| \leq \|\vec{g}(\vec{x})\|$ and as $\vec{x} \to \vec{a}$ the limit of $\vec{g}(\vec{x})$ exists and equals $\vec{0}$, then $\vec{f}(\vec{x}) \to \vec{0}$: Indeed, this follows from the definition.

Examples:

- Suppose $\lim_{x_1 \to a_1} g(x_1) = b$ where $g : \mathbb{R} \to \mathbb{R}$ is a function then $\lim_{\vec{x} \to \vec{a}} g(x_1)$ exists and equals b. Now $|g(x_1) - b| < \epsilon$ whenever $0 < |x_1 - a_1| < \delta$. Thus $|g(\vec{x}) - b| < \epsilon$ whenever $0 < |x_1 - a_1| \leq |\vec{x} - \vec{a}| < \delta$. So $\lim_{(x,y) \to (0,0)} x^2 = 0$.
- The limit $\lim_{(x,y) \to (0,0)} \frac{xy}{x^2+y^2}$ does NOT exist: Indeed, suppose it does and equals L. This means that $|f(x,y) - L| < \frac{1}{100}$ when $0 < \|(x,y)\| < \delta$. Thus if $y = x$ or $y = 2x$ and $0 < |x| < \frac{\delta}{\sqrt{5}}$, then $|f(x,y) - L| < \frac{1}{100}$. This means that $|\frac{1}{2} - L| < \frac{1}{100}$ and $|\frac{2}{5} - L| < \frac{1}{100}$ which is a contradiction. Note that individually, if y is fixed and $x \to 0$ or vice-versa, the limits exist and equal 0. So one way to prove that the limit does not exist is to compute it along two different paths and get different answers.
- The limit $\lim_{(x,y) \to (0,0)} \frac{x^2y^2}{x^2+y^2}$ exists and equals 0: Indeed, $|\frac{x^2y^2}{x^2+y^2}| \leq x^2$ which goes to 0.

The following theorem allows us to almost *directly* argue non-existence using different paths. To this end, we first define the convergence of sequences in the usual way. Theorem: Suppose $\lim_{(x,y)\to(a,b)} f(x,y)$ exists and equals L. If $x_n \to a$, $y_n \to b$ where (x_n, y_n) lie in the domain of f for all n, then $\lim_{n\to\infty} f(x_n, y_n)$ exists and equals L.