1 Recap

- Exterior points, boundary points, closed sets.
- Limits, continuity, Sandwich law, examples.
- By the way, limits are unique: If $\lim_{\vec{x}\to\vec{a}} \vec{f}(\vec{x}) = \vec{L}_1, \vec{L}_2$, then $\|L_1 - L_2\| \leq 2\epsilon$ for all $\epsilon > 0$.

2 Limits and continuity

Theorem: Suppose $\lim_{(x,y)\to(a,b)} f(x,y)$ exists and equals L. If $x_n \to a$, $y_n \to b$ where (x_n, y_n) lie in the domain of f for all n, then $\lim_{n\to\infty} f(x_n, y_n)$ exists and equals L.
Proof: Given $\epsilon > 0$ choose $\delta > 0$ such that whenever $0 < \|(x,y) - (a,b)\| < \delta$ and (x,y) lie in the domain of f, $|f(x,y) - L| < \epsilon$. Now choose N large enough so that whenever $n > N$, $|x_n - a| < \frac{\delta}{2}$ and $|y_n - b| < \frac{\delta}{2}$. Then $\|(x_n, y_n) - (a,b)\| < \delta$ and hence for all $n > N$, $|f(x_n, y_n) - L| < \epsilon$. $\square$
The above theorem makes the proof of *non-existence* much easier. The same theorem can be stated for more than two variables too. In fact, if the limits of $f(x_n, y_n)$ exist and are equal for *all* such convegent sequences $x_n \to a, y_n \to b$, then by contradiction, we can conclude that the limit of $f(x,y)$ exists in the multivariable sense (HW).

Limit and Continuity laws:

- Assume that $\vec{f}, \vec{g} : S \to \mathbb{R}^n$ are two functions.
- Suppose $\lim_{\vec{x}\to\vec{a}} \vec{f}(\vec{x}) = \vec{b}$ and $\lim_{\vec{x}\to\vec{a}} \vec{g}(\vec{x}) = \vec{c}$.
- $\lim_{\vec{x}\to\vec{a}}(\vec{f}(\vec{x}) + \vec{g}(\vec{x}))$ exists and equals $\vec{b} + \vec{c}$.
- $\lim_{\vec{x}\to\vec{a}} \lambda\vec{f}(\vec{x}) = \lambda\vec{b}$.
- $\lim_{\vec{x}\to\vec{a}}(\vec{f}(\vec{x}).\vec{g}(\vec{x}))$ exists and equals $\vec{b}.\vec{c}$.
- $\lim_{\vec{x}\to\vec{a}} \|\vec{f}(\vec{x})\|$ exists and equals $\|\vec{b}\|$.
- If $\vec{f}, \vec{g}$ are scalar-valued, $\vec{g}(\vec{x})$ is not zero in a neighbourhood of $\vec{a}$ (intersected with S), and $c \neq 0$, then $\lim_{\vec{x}\to\vec{a}} \frac{f}{g} = \frac{b}{c}$.
- The same laws hold for continuity too.

Proofs of 1,2,3:

- Choose $\delta > 0$ so small that whenever $0 < \|\vec{x} - \vec{a}\| < \delta$ and $\vec{x} \in S$, $\|\vec{f}(\vec{x}) - \vec{b}\| < \frac{\epsilon}{2}$ and $\|\vec{g}(\vec{x}) - \vec{c}\| < \frac{\epsilon}{2}$. Thus $\|\vec{f}(\vec{x}) + \vec{g}(\vec{x}) - \vec{b} - \vec{c}\| < \epsilon$ by the triangle inequality.
- Without loss of generality assume that $\lambda \neq 0$ (why?). Choose $\delta > 0$ so small that whenever $0 < \|\vec{x} - \vec{a}\| < \delta$ and $\vec{x} \in S$, $\|\vec{f}(\vec{x}) - \vec{b}\| < \frac{\epsilon}{|\lambda|}$. Thus we are done.

- Let $\vec{f}(\vec{x}) - \vec{b} = \vec{h}_1$ and $\vec{g}(\vec{x}) - \vec{c} = \vec{h}_2$. $|\vec{f}(\vec{x}).\vec{g}(\vec{x})) - \vec{b}.\vec{c}| \leq |(\vec{h}_1 + \vec{b}).(\vec{h}_2 + \vec{c}) - \vec{b}.\vec{c}|$. Now we use the triangle inequality to see that it is less than $|\vec{h}_1.\vec{h}_2| + |\vec{h}_1.\vec{c}| + |\vec{h}_2.\vec{b}|$. By the Cauchy-Schwarz inequality it is less than $\|\vec{h}_1\|\|\vec{h}_2\| + \|\vec{h}_1\|\|\vec{c}\| + \|\vec{h}_2\|\|\vec{b}\|$. We want to make each term less than $\frac{\epsilon}{3}$ by choosing δ small enough. This can be done for the last two terms almost by assumption. For the first term, if necessary, shrink δ so that $\|\vec{h}_i\| < \frac{\sqrt{\epsilon}}{\sqrt{3}}$.

Proofs of 4,5:

- $|\|\vec{f}(\vec{x})\| - \|\vec{b}\|| \leq \|\vec{f}(\vec{x}) - \vec{b}\|$ by the triangle inequality. We are done.
- We may assume that $f = 1$ without loss of generality (why?). $|\frac{1}{g} - \frac{1}{c}| = \frac{|g(\vec{x})-c|}{|c||g(\vec{x})|}$. Choose δ so small that $|g(\vec{x}) - c| < \frac{|c|}{2}$. Thus $\frac{|c|}{2} < |g(\vec{x})| < \frac{3|c|}{2}$. Thus $|\frac{1}{g} - \frac{1}{c}| < \frac{2|g(\vec{x})-c|}{c^2} < \epsilon$ when δ is even smaller.
- We can also prove that (HW) if $\lambda(x)$ is continuous and $\vec{f}$ is so then so is $\lambda(x)\vec{f}$.

Examples:

- Another way to prove that the components of a continuous vector-valued function are continuous is by noting that they are dot products. The converse follows from the above properties.
- Since the components are continuous the identity function is so as well.
- Linear maps are continuous: $\vec{f}(\vec{a} + \vec{h}) = \vec{f}(\vec{a}) + \vec{f}(\vec{h}) = \vec{f}(\vec{a}) + \sum_i h_i f(\vec{e}_i)$ which goes to $\vec{f}(\vec{a})$ as $\vec{h} \to \vec{0}$.
- By induction, polynomials are continuous.
- The above properties imply that rational functions are continuous away from the zeroes of their denominator provided we prove that if $g(\vec{a}) \neq 0$ then it is so in a neighbourhood. But this follows from continuity of $g(\vec{x})$. (Actually, as we saw, this assumption is superfluous.)

Composition:

Theorem: Let $\vec{g} : U \subset \mathbb{R}^m \to \mathbb{R}^n$ be a function that is continuous at $\vec{a} \in U$. Let $\vec{g}(U) \subset V \subset \mathbb{R}^n$ and let $\vec{f} : V \subset \mathbb{R}^n \to \mathbb{R}^p$ be a function that is continuous at $\vec{g}(\vec{a}) \in V$. Then $\vec{f} \circ \vec{g} : U \to \mathbb{R}^p$ is continuous at $\vec{a}$.