Notes for 10th Jan (Thursday)

1 The road so far...

1. Discussed complete induction.
2. Pigeon-hole principle and examples.

2 Pigeon hole, permutations and combinations

Example 1.7 in walk through combinatorics (illustrates the importance of an algebraic structure called a graph).

Permutations and combinations are expected to be known and we will not focus too much on them. However, for the sake of completeness (and to illustrate some general principles), the number of bijections of $[n] = \{1, 2, \ldots, n\}$ to itself is called the number of permutations of n letters.

Lemma 2.1. *The number of permutations of n letters is $P_n = n!$.*

Proof. Before we prove this lemma, here are a couple of important principles (part of your HW) :

1. (Sum principle) If there are m ways of doing A and n ways of doing B where A and B are disjoint, then there are $m + n$ ways of doing either A or B.
2. (Product principle) If A and B are "independent", then there are mn ways of doing A and B. More rigorously, the size of a union of m disjoint sets, each of size n is mn. Alternatively, $|A \times B| = |A||B|$.

Trivially, $P_1 = 1$. Assume that $P_n = n!$. There are $n + 1$ possible images of $n + 1$. The product principle states that $P_{n+1} = (n+1)P_n = (n+1)!$. (Make this more rigorous.) $\square$

A k-permutation is defined as a $1-1$ function from $[k]$ to $[n]$. (It is an ordered list of k elements). A k-combination (or a choice of k-elements out of n elements) is a k-element subset of $[n]$.

Lemma 2.2. *The number of k-permutations is $P(n, k) = n(n-1)\ldots(n-k+1) = \frac{n!}{(n-k)!}$.*

Proof. Intuitively, the first element can be chosen in n ways, the second in $n - 1$ ways and so on. By the product principle we get the answer. More rigorously, take a k-list and an $n - k$ permutation of the remaining $n - k$ elements in $[n]$. This way we get a permutation of $[n]$. This map is a bijection and hence $P(n, k)(n - k)! = n!$. $\square$

Lemma 2.3. *The number of k-combinations is $\binom{n}{k}$.*

Proof. Define an equivalence relation between k-lists as being equivalent if they have the same elements (the set equality relation). Every equivalence class has $k!$ elements (permutations). Since this is just the set equality relation, the number of classes $C(n,k)$ is the number of k-combinations. The number of k-lists is $P(n,k)$. Hence, $C(n,k) = \frac{P(n,k)}{k!}$. □

The above proof illustrates an important principle of exploiting symmetry - The quotient principle : If there is a symmetry on the set A such that A is partitioned into k equal-sized pieces of size n where the symmetry operation takes each piece to itself, then $|A| = kn$. Here are examples :

1. A gardener has five red flowers, three yellow flowers, and two white flowers to plant in a row. In how many ways can she do that ?
   First consider all the flowers to be different. The number of ways to plant in a row is the number of permutations which is 10!. We define an equivalence relation among the plantings by identifying flowers of the same colour. Then the number of equivalence classes is what we want. That number $\times classes = 10!$. In each equivalence class, we simply permute all the flowers having the same colour. There are 5!3!2! ways of doing that (product principle).

2. A multiset is a set where repetition is allowed to occur (it is simply a set with an equivalence relation basically). It can be specified by saying how many times each of its elements occurs. For instance $\{\{f,o,o,r\}\} = \{\{o,f,o,r\}\}$ is a 4-element multiset whose underlying set is $\{f,o,r\}$. It can be specified by simply the numbers $1 \leq 1 \leq 2$ (corresponding to the number of times f,r,o occur in the multiset respectively).
   A "real-life" example of a k-multiset is handing k identical apples to n children or to place k identical books on n distinct shelves. Actually, this problem can be looked at in two ways :

   (a) Counting solutions to $m_1 + m_2 + \ldots + m_n = k$ where $m_i \geq 0$ is the number of books on the i^{th} shelf.

   (b) From the perspective of the books, each book can "choose" to be placed on any of the shelves. So record the "shelf number" (or the name of the child to which the apple is handed out) for each of the books. Clearly, there can be repetitions among the shelf numbers. Since we do not distinguish between the books, this is a multiset. Indeed, if we place 2 books on the first shelf, 1 on the second and 1 on the third (let's say there are 3 shelves), then the multiset of shelf numbers is $\{\{1,1,2,3\}\}$. (The order does not matter because the books are identical.)

   The number of k-element multisets from $[n]$ is $C(n+k-1,k)$. Indeed, this is simply the number of tuples $(x_1, x_2, \ldots, x_k)$ where $n \geq x_i \geq 1$ and $x_1 \leq x_2 \leq x_3 \ldots$. There is a bijection from this to k element subsets of $[n-k+1]$. Indeed, take $(x_1, \ldots, x_n) \to (x_1, x_2+1, x_3+2, \ldots)$.