Notes for 16th Jan (Wednesday)

1 The road so far...

1. More about multisets.
2. Defined cycles and stated the cyclic decomposition theorem of permutations. Defined groups and gave examples and non-examples.

2 Cycles in permutations

Theorem 1. *Let $a_1, \ldots, a_n \geq 0 \in \mathbb{N}$ satisfy $\sum ia_i = n$. Then the number of elements in S_n with a_i cycles of length i is $\frac{n!}{a_1!\ldots a_n! 1^{a_1} 2^{a_2} \ldots}$.*

Proof. The answer suggests that the quotient principle is in play. Clearly we need to impose an equivalence relation on the permutations. First, we write each permutation as simply $1, 2 \ldots, n$ written in some order (there are $n!$ ways to do this). Then, we simply insert parantheses of size i, a_i times. This gives a cycle decomposition with nondecreasing cycle lengths. We define an equivalence relation by identifying two cycle decompositions if they correspond to the same permutation. Of course, the only freedom lies in permuting the order of the a_i cycles (done in $a_i!$ ways) and in cyclically permuting the order within each i-cycle (done in i^{a_i} ways by the product principle). Hence, the product principle says that this can be done in $\Pi_i a_i! i^{a_i}$ ways. □

3 Ordinary Generating Functions

The Fibonacci sequence is defined by a recurrence relation $a_n = a_{n-1} + a_{n-2}$. One way to "solve" this recurrence relation is by finding an efficient algorithm to calculate it. (Do it as an exercise in C++.) However, ideally we would like an explicit formula. A much better way to do this is using generating functions : Let $\{a_n\}$ be a sequence of real numbers. Then the formal power series $\displaystyle\sum_{n=0}^{\infty} a_n x^n$ is called the ordinary generating function (ogf) of the sequence.

A formal power series (fps) is defined purely using its coefficients and has no meaning as such (it need not be the Taylor series of any function for instance). Here are some definitions/properties of fps $f(x) = \sum a_n x^n, g(x) = \sum b_n x^n, h(x) = \sum c_n x^n$:

1. Def : $f(x) := g(x)$ if $a_n = b_n \forall n$.

2. Def : $f(x) + g(x) := \sum(a_n + b_n)x^n$ and $cf(x) := \sum c a_n x^n$.

3. Def : $f(x)g(x) := \sum_{n=0}^{\infty} \sum_{k=0}^{n} a_k b_{n-k} x^n$.

4. Property : $f(x)g(x) = g(x)f(x)$, $(f(x)g(x))h(x) = f(x)(g(x)h(x))$, $(f(x)+g(x))+h(x) = f(x)+(g(x)+h(x))$, $f(x)+g(x) = g(x)+f(x)$, and $xf(x) = \sum_{n=0}^{\infty} a_n x^{n+1} = \sum_{n=1}^{\infty} a_{n-1} x^n$.

5. Def : f is said to have a multiplicative inverse (denoted by $\frac{1}{f}$) if $f.\frac{1}{f} = 1$. Property : Multiplicative inverses are unique.

6. Property : $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$.

7. Def : $f'(x) := \sum n a_n x^{n-1}$, $\int f(x)dx := \sum \frac{a_n x^{n+1}}{n+1}$.

Write the ogf of the Fibonacci sequence as $f(x) = x + x^2 + 2x^3 + \ldots$. Now, one can "shift" the coefficients of an fps by simply multiplying by a power of x. That is, $xf(x) = \displaystyle\sum_{n=1}^{\infty} a_n x^{n+1} = \sum_{n=2}^{\infty} a_{n-1} x^n$ and likewise, $x^2 f(x) = \sum_{n=3}^{\infty} a_{n-2} x^n$. Therefore, the Fibonacci relation is $f(x) = (x+x^2)f(x)+x$. Hence, $f(x) = \frac{x}{1-x-x^2}$. One can simplify this expression using partial fractions as $f(x) = \frac{-1}{\phi+1/\phi}\left(\frac{\phi}{x+\phi} + \frac{1/\phi}{x-1/\phi}\right) = \frac{1}{\sqrt{5}} \displaystyle\sum_{n=0}^{\infty} x^n(\phi^n - (-1)^n \frac{1}{\phi^n})$.