Notes for 17th Jan (Thursday)

1 The road so far...

1. Counted the number of permutations with a given cycle structure.
2. Defined Ordinary Generating Functions and discussed the properties of formal power series.

2 Ordinary Generating Functions

Likewise, here is another example : A frog population grows fourfold each year. On the first day of each year, 100 frogs are taken out of the lake and shipped into another. If there were 50 frogs to begin with, find the number after n years. So $a_n = 4a_{n-1} - 100$ with $a_0 = 50$. So the ogf is $f(x) = \sum a_n x^n$. Now the recurrence relation can be represented as $f(x) - a_0 = 4xf(x) - \sum_{n=0}^{\infty} 100x^{n+1}$ which means that $f(x) = \frac{a_0 - 100x/(1-x)}{1-4x}$ (again use partial fractions) to get $a_n = 50.4^n - 100\frac{4^n-1}{3}$.

As mentioned earlier, products of generating functions is defined using the Cauchy product. (This definition coincides with that of multiplying actual series as you will see in your analysis class.) Here is a very useful observation : Suppose $f(x)$ is a generating function. Then $f(x)\frac{1}{1-x}$ is the generating function of $a_0, a_0 + a_1, a_0 + a_1 + a_2, \ldots$ (that is the partial sums!). For instance, suppose $H_n = 1 + \frac{1}{2} + \ldots + \frac{1}{n}$ (the harmonic numbers). We can prove the following nice relation between them : $\sum_{k=1}^{n} H_k = (n+1)(H_{n+1} - 1)$. Indeed, let $a_n = \frac{1}{n}$ and $f(x) = \sum_{n=1}^{\infty} a_n x^n$. Then $\frac{f(x)}{1-x} = \sum_{n=1}^{\infty} H_n x^n$. Therefore, $\frac{f(x)}{(1-x)^2} = \sum_{n=1}^{\infty} (\sum_k H_k) x^n$. Now, $\frac{1}{(1-x)^2} = \sum (n+1)x^n$. Hence, upon multiplication, $\sum_k H_k = \sum \frac{1}{k}(n+1-k) = (n+1)H_n - n$.

Another combinatorial consequence of products is

Lemma 2.1. *Let a_n be the number of ways to build a certain structure on an n-element set, and let b_n be the number of ways to build another structure on an n-element set. Let c_n be the number of ways to partition n into $S = \{1, 2, \ldots, i\}$ and $T = \{i+1, \ldots, n\}$ (they can be empty) and build a structure of the first kind on S and the second kind on T. Then $A(x)B(x) = C(x)$.*

Proof. Suppose we split it into $(i, n-i)$. Then by the product principle, the number of ways is $a_i b_{n-i}$. By the sum principle, $c_n = \sum_{i=0}^{n} a_i b_{n-1}$. □

Discussion of examples 8.6, 8.7, 8.8.