# Notes for 22nd Jan (Tuesday)

## 1 The road so far...

1. Ogfs and a combinatorial lemma.

## 2 Exponential Generating Functions

An ogf is unlikely to solve a problem where the sequence grows too fast. So we make the following definition : If $a_n$ is a sequence, then the formal power series defined by $F(x) = \sum_{n=0}^{\infty} \frac{a_n}{n!} x^n$. Here are some examples :

1. $a_0 = 1, a_{n+1} = (n+1)(a_n - n + 1)$. Let $F(x)$be the egf. Then

$$F(x) - 1 = \sum_{n=0} \frac{a_{n+1}}{(n+1)!} x^{n+1} = \sum_{n=0} (\frac{a_n}{n!} x^{n+1} - \frac{n-1}{n!} x^{n+1}) = xF(x) + xe^x - x^2 e^x$$

$$\Rightarrow F(x) = \frac{1}{1-x} + xe^x \tag{1}$$

   Therefore $a_n = n! + n$.

2. $a_0 = 0, a_{n+1} = 2(n+1)a_n + (n+1)!$.

$$F(x) - 0 = 2\sum_{n=0} \frac{a_n x^{n+1}}{n!} + \sum_{n=0} x^{n+1} = 2xF(x) + \frac{x}{1-x}$$

$$\Rightarrow F(x) = \frac{x}{(1-x)(1-2x)} = \sum (2^n - 1)x^n. \tag{2}$$

Here is another combinatorial lemma.

**Lemma 2.1.** *Let $a_n, b_n$ be the number of ways to build two structures on $[n]$. Let $c_n$ be the number of ways to separate $[n]$ into two disjoint subsets and put one structure on the first and the second on the other. Let $A(x), B(x), C(x)$ be the e.g.fs. Then $C(x) = A(x)B(x)$.*

*Proof.* Suppose we split it into $(k, n-k)$. This can be done in $\binom{n}{k}$. Then $c_n = \sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k}$. Now $A(x)B(x) = \sum_{n=0} \sum_{k=0}^{n} \frac{a_k b_{n-k}}{k!(n-k)!} x^n = \sum_n \sum_k \binom{n}{k} a_k b_{n-k} \frac{x^n}{n!} = C(x)$. $\square$

Here is an example to apply this lemma : A cricket coach has n players to work with. He splits them into two groups and asks members of each group to form a line. Then he asks each member to take on orange shirt, or a white shirt, or a red shirt. Members of the other group keep their blue shirt. In how many ways can this happen ?
$a_k = 3^k k!$, $b_k = k!$. So $A(x) = \sum 3^n x^n = \frac{1}{1-3x}$ and $B(x) = \sum x^n = \frac{1}{1-x}$. So $C(x) = \frac{1}{(1-3x)(1-x)} = \frac{1}{2}(\frac{3}{1-3x} - \frac{1}{1-x})$. Therefore $c_n = n! \frac{3^{n+1}-1}{2}$.

# 3 Graphs

Euler solved the bridges of Könisberg problem by inventing graphs. (Picture drawn on the board). Graphs are very important not just for applications, but also in pure mathematics (functional analysis (Cadison-Singer conjecture), algebra (Cayley graphs of groupsm classification of Lie algebras), topology (simplest examples of CW complexes), differential geometry (harmonic maps), algebraic geometry (stable maps)), and physics (QFT).
Definition : A graph $G$ is a finite set $V$ (called vertices) and a multiset $E$ consisting of two element multisets from $V$ (called edges). The number of edges connected to a vertex $v$ is called the degree of $v$. (Loops are counted twice.) If there are no multiple edges or loops, such a graph is said to be a simple graph. A subgraph $S \subset G$ is a graph whose vertices form a subset of $V$ and whose edges form a subset of $E$.

**Lemma 3.1.** *(Handshaking lemma)* $\sum_{v \in V} deg(v) = 2|E|$, *i.e., in party an even number of people must have shaken an odd number of hands.*

*Proof.* We prove this assuming $G$ is simple. The general case will be given as a HW. Define an equivalence relation on the set of ordered pairs $(i, j)$ such that $\{i, j\} \in E$ saying that $(i, j) \sim (j, i)$ and $(i, j) \sim (i, j)$. The number of ordered pairs is $\sum_{v \in V} deg(v)$ (we fix $i$ and count all the ordered pairs with that as the first coordinate, and so on). Clearly the number of equivalence classes is $|E|$. Hence we are done. $\square$