Notes for 23rd Jan (Wednesday)

1 The road so far...

1. EGFs and a combinatorial lemma.
2. Defined graphs (actually multigraphs), subgraphs, and degree. Proved the handshaking lemma for simple graphs.

2 Graphs

A trail is a sequence of distinct edges such the endpoint of e_i is the starting point of e_{i+1}. If the edges are not distinct the trail is called a walk. If a trail starts and ends at the same vertex, it is called a closed trail. If a trail uses all edges then we call it an Eulerian trail. If a trail does not touch a vertex twice, it is called a path or sometimes, an arc.
If any two vertices can be connected by a path then such a graph is called connected. Examples will be drawn on the board.

There are many ways to represent graphs, i.e., tell a computer about a graph. One way is an adjacency list, i.e., an array of lists like $a[1] = (1,2),(1,3), a[2] = (2,4), \ldots$. Another is an adjacency matrix : $A_{ij} = \textit{number of edges}$ if (i,j) is an edge and 0 otherwise (a loop is counted as 1 edge -¿ Caution : For degree, we count it as two). Note that since we are considering *undirected* graphs, $A_{ij} = A_{ji}$. An adjacency matrix is very useful to do some things - For instance,

Lemma 2.1. *The number of walks (NOT paths or trails) of length n (that is number of edges traversed is n) between i and j is $(A^n)_{ij}$*

Proof. Indeed, for $n = 1$ it is trivial. Assuming truth for $1, 2 \ldots, n$, then an $n+1$-length walk between i and j comes from n-length walks to the d_j neighbours of j. It equals $\sum_{k|A_{jk}=1} (A^n)_{ik} = \sum_k (A^n)_{ik} A_{kj} = (A^{n+1})_{ij}$. □

Hence, the distance (the smallest length of a trail) between i and j is the smallest n such that $(A^n)_{ij} > 0$. In particular, A can be used to decide whether a graph is connected or not. (Calculate $A^{|E|}$ and check if all non-diagonal entries are positive or not.)

Lemma 2.2. *Every graph is a finite disjoint union of connected subgraphs, i.e., there are no edges between the subgraphs and no vertices in common. These subgraphs are called connected components.*

Proof. Declare an equivalence relation between vertices : $i \sim i$ and $i \sim j$ if i and j can be connected by a path. (Why is this an equivalence relation ?) This relation partitions the vertex set into finitely many subsets $V_1, \ldots, V_k$. Within each equivalence class, the vertices are connected. There are no edges between different equivalence classes. Hence V_i along with the edges between vertices is a connected subgraph G_i. $G = \cup_i G_i$. □