Notes for 24th Jan (Thursday)

1 The road so far...

1. Defined walks, trails, closed trails, closed Eulerian trails, and paths.
2. Adjacency lists and matrices. Proved a formula for finding the number of length-n walks.
3. Proved that every graph is a union of connected components.

2 Graphs

When can we say two graphs are secretly the same ? Def : A graph isomorphism $f : G \to H$ is a bijection between the vertex sets $\tilde{f} : V(G) \to V(H)$ such that $(\tilde{f}(i), \tilde{f}(j)) \in E(H)$ if and only if $(i, j) \in E(G)$. (Please think about how to make this definition rigorous. Hint : A multiset is simply a set with an equivalence relation.)
This is a hard problem for even a computer to solve. A property that is invariant under graph isomorphism is the number of connected components.

Here is an application of the handshaking lemma.

Lemma 2.1. *If a graph has exactly two vertices of odd degree, there must a path between these vertices.*

Proof. Suppose there is no such path. Then the vertices belong to different connected components. Applying the handshaking lemma to each of these components we run into a contradiction. □

Now we solve the Könisberg problem of determining when graphs have Eulerian cycles.

Theorem 1. *A connected graph G (even one that has multiple edges and loops) has a closed Eulerian trail if and only if all vertices of G have even degree.*

Proof. If there is a closed Eulerian trail $v_1e_1v_2e_2v_3 \dots v_ne_nv_1$, then roughly speaking, we enter a vertex as many times as we leave it. Hence the degree of every vertex is even. More rigorously, we induct on the number of edges. Firstly we may assume there are no loops (they contribute as even numbers to the degrees anyway). Assume that there are at least two edges. Suppose we delete e_1 and e_n. Either the graph is disconnected (in which case we are done), or it is not and we can join v_2 and v_n by an edge $\tilde{e}$. Then we get a closed Eulerian trail in a smaller graph starting at v_2. (It is a cycle because by connectedness there was already a trail starting at v_2 ending at v_n.) The induction hypothesis implies that all vertices of the smaller graph have even degree. But we changed the parity of the degrees in an even manner and hence the original graph also has even degrees.

Suppose all vertices have even degree and there are no loops w.log. Choose a vertex v. Delete two edges e_1, e_2 (with endpoints v_1, v_2). Assume that the resulting graph is connected (what happens when it is disconnected will be given as a HW). Join v_1 and v_2 with an edge $\tilde{e}$. The resulting graph still has even degrees and fewer edges and hence has a closed Eulerian trail denoted as $v_1 C v_1$ starting and ending at v_1. Now $\tilde{e}$ is traversed either from v_1 to v_2 or from v_2 to v_1. If the former happens, by a permutation, let v_2 be the starting vertex. So without loss of generality, e is traversed from v_2 to v_1. Delete e and traverse instead from v_2 to v by e_2 and then from e to v_1 by e_1. This closed trail is Eulerian. □

We have the following corollary about Eulerian trails (not closed).

Lemma 2.2. *Let G be a connected graph. Then G has an Eulerian trail starting at S and ending at a different vertex T iff S and T have odd degree and the other vertices have even degree.*

Proof. Connect S and T by an edge $\tilde{e}$. If G has an Eulerian trail from S to T, the new graph has an Eulerian cycle and hence the claim. If, on the other hand, S and T have odd degree, then new graph has a closed Eulerian trail starting at S. Delete $\tilde{e}$ to get the desired Eulerian trail. □

Akin to closed Eulerian trails (sometimes called Eulerian cycles), a Hamiltonian cycle is defined as a closed path that passes through every vertex exactly once (likewise for a Hamiltonian path). This notion is useful to answer questions like "In a party where guests are seated at a round table, is there an arrangement so that every guest knows both people seated next to them ?" This is computationally very hard (an NP-complete problem). Nonetheless, here is still an interesting result.

Theorem 2. *Let $n \geq 3$ and G be a simple graph on n vertices. Assume all vertices are of degree at least $\frac{n}{2}$. Then G has a Hamiltonian cycle.*