Notes for 29th Jan (Tuesday)

1 The road so far...

1. Proved that Eulerian tours exist if and only if the degree of every vertex is even.
2. Corrected a statement made earlier ($(A^n)_{ij}$ is the number of *walks* between i and j).
3. Defined Hamiltonian cycles.

2 Trees

A tree is defined as a connected simple graph that does not contain cycles, i.e., simple closed trails. Here is an important alternate characterisation.

Theorem 1. *A connected simple graph G is a tree iff it is minimally connected, i.e, if we remove any edge of G it becomes disconnected.*

Proof. If there is a cycle in G $v_1e_1v_2e_2\ldots, v_ne_nv_1$, then removing e_n cannot disconnect G because there is another path from v_1 to v_n.
If G is not minimally connected, i.e., there are vertices v_1, v_2 with an edge e between them such that removing e does not disconnect G, i.e., there is another path between v_1 and v_2, then we have a cycle in G. □

As a corollary

Lemma 2.1. *A connected simple graph is a tree iff for any pair of vertices x, y there is exactly one path from x to y.*

Proof. If for each pair, there is only one path, then G is minimally connected. Indeed, if not, then removing an edge between v_1, v_2 still leaves a path between v_1 and v_2. A contradiction.
If G is a tree but there are two paths P and Q joining x, y. Take the symmetric difference of P and Q, i.e., edges that are part of either P xor Q. This symmetric difference is a union of cycles (will be a part of your HW). □

How does one construct trees ?

Theorem 2. *All trees on n vertices have $n-1$ edges. Conversely, all connected graphs on n vertices with exactly $n-1$ edges are trees.*

Proof. We need the following lemma.

Lemma 2.2. *Let T be a tree on n vertices where $n \geq 2$. Then T has at least two vertices whose degree is* 1*. (Such a vertex is called a leaf.)*

Proof. For $n = 2$, there has to be only one edge by minimal connectedness. Suppose this statement is true for $1, 2 \ldots, n$. Then remove one vertex (and the edges incident on it) that does not have degree 1. (If all the vertices have degree one, we are done.) The resulting disconnected graph's components are all trees. (Indeed if they had cycles, so would the original graph.) So there at least two vertices in each component of degree 1. At least one of the leaves of each component cannot be connected to the removed vertex by an edge (otherwise we will have a cycle). Since there are at least two components, we are done. □

Now we prove the main theorem by induction.
For $n = 1$, there are no loops (by definition trees are simple graphs). If there is exactly one edge it is a tree.
Assuming truth for $1, 2 \ldots, n$, remove a leaf (and its edge) from the tree. The resulting graph has $n - 1$ edges. Since we removed only one edge, the original tree had n edges. Conversely, if there are n edges, by the handshaking lemma, at least two vertices have degree 1. Remove one of these vertices and the corresponding edge. The resulting graph is still simple, connected, has n vertices and $n - 1$ edges, and is hence a tree. Adding a degree 1 vertex still keeps it minimally connected and hence it is a tree. □

By the way, a disjoint union of trees is called a forest. It is easy to see that a forest on n vertices with k connected components has $n - k$ edges.