# Notes for 30th Jan (Wednesday)

## 1 The road so far...

1. Siddhartha did some symmetric differences, trees (characterisation in terms of number of edges, and stated the Euler formula.

## 2 Graphs

**Theorem 1.** *Let $n \geq 3$ and $G$ be a simple graph on $n$ vertices. Assume all vertices are of degree at least $\frac{n}{2}$. Then $G$ has a Hamiltonian cycle.*

*Proof.* Assume that $G$ does not have such a cycle. Consider all the graphs with the same vertex set as $G$. Since the complete graph on $|V_G|$ vertices (the graph where every vertex is connected to every other vertex) has a Hamiltonian cycle, there exists a non-Hamiltonian graph $H$ containing $G$ such that addition of any edge produces a Hamiltonian cycle.

Let $x, y \in V(H)$ such that $(x, y) \notin E(H)$. Since the edge $(x, y)$ if added to $H$ creates a Hamiltonian cycle, $G^{'}$ has a Hamiltonian path $x, z_1, z_2, \ldots, z_n = y$. $x$ and $y$ together have at least $n$ neighbours. Therefore, PHP implies that there is a $z_i$ so that $(x, z_i), (z_{i-1}, y) \in E(H)$. This is a contradiction because $xz_2 \ldots z_{i-1}yz_{n-1} \ldots z_i$ is a Hamiltonian cycle. $\square$

## 3 Planarity

Suppose a farming community has 3 houses and 3 wells. The families cannot stand each other and prefer not to meet when they walk to the wells. Can we build roads from the houses to the wells so that the 9 roads do not intersect ? Turns out that one cannot.
Here are relevant definitions : A graph $G$ is said to be embedded in $\mathbb{R}^n$ if there are 1-1 maps $f : V_G \to \mathbb{R}^n$ and $g : E_G \to$ *continuous paths in* $\mathbb{R}^n$ such that $f(i), f(j)$ are connected by the path $g(\{\{i, j\}\})$ and the paths in the image of $g$ do not intersect in any points other than possibly their endpoints, i.e., the vertices are points in $\mathbb{R}^n$ and the edges are continuous paths between the vertices such that the edges do not intersect anywhere other than possibly their endpoints.
Here is another : A graph $G$ is said to be planar if it can be embedded in $\mathbb{R}^2$. Note that planarity is independent of graph isomorphism.
Finally : Removing the edges of a planar graph embedded in a plane, we get a bunch of "disconnected pieces" that we call faces.
We have Euler's formula :

**Theorem 2.** *Let $G$ be a connected planar graphs. Then $V - E + F = 2$. (The number $V - E + F$ is called the Euler characteristic of the graph and plays an important role in topology.)*

*Proof.* We induct on $E$. If $E = 1$, then $V = 2$ and "hence" $F = 1$ or $V = 1$ and $F = 2$. (It is not trivial to justify these things rigorously and so we will not attempt to do so. The key phrase is "Jordan curve theorem".) Assume truth for $1, 2, \ldots, E - 1$ edges. Now we have two possibilities :

- If we can delete an edge $e$ such that the new graph $G'$ is still connected, then $e$ is in a cycle and therefore divides the plane into two faces. $G'$ has $E - 1$ edges, $V$ vertices, and $F - 1$ faces. Hence $V - E + F = 2$.

- If there no such edge, then $G$ is a tree. So $V = E + 1$. But $F = 1$ because there are no cycles. Hence we are done.

□

Here is an application of the above result : The graph $K_{3,3}$ consisting of three wells and three houses as above is not planar. Indeed, according to Euler, it has 5 faces. Firstly, there are no triangles in the graph (why?) Secondly, every face needs to bound at least 3 edges (this is not trivial to justify again). So every face has at least 4 edges. But each edge is used in at most two different faces and hence we need 10 edges. But we only have 9.

Another application is to prove that the graph $K_5$ where all vertices are connected by edges is not planar : Indeed, we must have seven faces by Euler. Each of the seven faces must at least have three edges and hence would need 21 edges which is an overcount. But we have only 10 edges anyway.

It turns out (Kuratowski and Wagner theorems) that every nonplanar graph secretly hides $K_{3,3}$ or $K_5$ in some way.

# 4 Number theory - The basics

Recall that primes are numbers (other than 1) that do not have any non-trivial factors. We proved that every natural is divisible by a prime.