Notes for 9th Jan (Wednesday)

1 The road so far...

1. Constructed integers and discussed their usual properties.
2. Constructed rationals out of integers.
3. Discussed Pythagorean triples.

2 Induction

Induction is actually one of the axioms for defining natural numbers :

If K is a set such that $0 \in K$, and for every natural n, whenever $n \in K$, then so is $n+1$, then $K = \mathbb{N}$. (This principle can be restated as suppose $n_0 \in K$, and the rest, then $n_0, n_1 + 1, \ldots = K$.)

Actually this statement can be proven to be equivalent to a stronger principle of induction : If K is a set such that $0 \in K$, and for every natural n, whenever $0, 1, 2 \ldots, n \in K$ then so is $n+1$, then $K = \mathbb{N}$.

The problem with induction as a proof technique is that one should know the “correct” theorem one is trying to prove beforehand, i.e., one cannot discover identities using induction. Induction is to be thought of as akin to recursion in computer programming. Here are some examples.

1. Define the Fibonacci numbers recursively (ZFC+Peano axioms allow recursive definitions) as $a_0 = a_1 = 1$, $a_n = a_{n-1} + a_{n-2}$. Prove that $a_n = \frac{\phi^n - (1-\phi)^n}{\sqrt{5}}$ $\forall\ n \geq 1$ where $\phi = \frac{1+\sqrt{5}}{2}$ (the so-called Golden ratio). By the way, from now onwards, we shall assume that we know all about real numbers.
   Of course, $a_1 = 1$. If the formula holds for n, then observing that $\phi^2 = \phi + 1$ we see that
   $$a_{n+1} = a_n + a_{n-1} = \frac{\phi^n - (1-\phi)^n + \phi^{n-1} - (1-\phi)^{n-1}}{\sqrt{5}} = \frac{\phi^{n+1} - (1-\phi)^{n+1}}{\sqrt{5}}$$
   which is the correct formula.
   The way one discovers this formula is far more interesting. (We will do that later on.)

2. The Binomial theorem : $(x+y)^n = \sum_{r=0}^{n} \binom{n}{r} x^r y^{n-r}$. The case of $n=0$ is trivial. Assume that the statement for n. Then

$$\begin{aligned}
(x+y)^{n+1} = (x+y)\sum_{r=0}^{n}\binom{n}{r}x^r y^{n-r} &= \sum_{r=0}^{n}\binom{n}{r}x^{r+1}y^{n-r} + \sum_{r=0}^{n}\binom{n}{r}x^r y^{n-r+1} \\
&= \sum_{r=1}^{n+1}\binom{n}{r-1}x^r y^{n+1-r} + \sum_{r=0}^{n}\binom{n}{r}x^r y^{n-r+1} \\
&= y^{n+1} + x^{n+1} + \sum_{r=1}^{n}\left(\binom{n}{r-1}+\binom{n}{r}\right)x^r y^{n+1-r} \qquad (1)
\end{aligned}$$

One can that $\binom{n+1}{r} = \binom{n}{r-1} + \binom{n}{r}$ for $n \geq 1, 1 \leq r \leq n$. Indeed,

$$\binom{n}{r-1} + \binom{n}{r} = \frac{n!}{(n-r)!(r-1)!}\left(\frac{1}{n-r+1} + \frac{1}{r}\right) = \binom{n+1}{r}.$$

3. The number 8 divides $3^{2n}-1$ for all $n \geq 0$. Indeed, for $n=0$ it is trivial. Assuming truth for n, $3^{2n+2}-1 = 3^{2n}9 - 1 = (3^{2n}-1)9 + 9 - 1$ which is divisible by 8.

4. Define a prime number $p \geq 2$ to be one that does not factor into a product of two strictly smaller natural numbers. Every natural $n \geq 2$ is divisible by a prime.
   Note that 2 is a prime and hence the base case is trivial. If we assume truth for $2, 3, \ldots, n$, then if $n+1$ is a prime it is of course divisible by a prime. If not, $n+1 = ab$. Since $a, b \leq n$, they are divisible by primes and hence so is $n+1$.

5. (Induction is useful for geometric problems too) Let $f(m)$ be the maximum number of domains into which m straight lines can divide the plane. Then $f(m) = \frac{m(m+1)}{2} + 1$.
   Indeed, for $m=0, m=1$, the answer is trivial. Assuming truth for $0, 1, 2 \ldots, m$, we shall prove it for $m+1$. Denote by $t_1, \ldots, t_k$ the $(m+1)^{th}$ straight line s crosses in order. Note that $k \leq m$. Since it cuts through a domain before and after crossing every line, it cuts the plane into $k+1$ more domains. Hence, $f(m+1) \leq f(m)+m+1$ with equality holding iff s crosses all the lines. Hence, $f(m+1) = \frac{m(m+1)}{2} + 1 + m + 1 = \frac{(m+1)(m+2)}{2} + 1$.

3 Pigeon hole, permutations and combinations

The following seemingly stupid result (dubbed the pigeon hole principle) is actually shockingly useful.

Theorem 1. *Let n, r, k be positive integers and $n > rk$. Suppose we have to place n identical balls into k identical boxes. Then there will be at least one box in which place at least $r+1$ balls.*

Proof. Suppose there is no such box. Then in k boxes we have at most r balls. Therefore, the total number of balls is $rk < n$. A contradiction. □

Here are some cool applications.

1. Five points are situated inside an equilateral triangle whose side has length one unit. Show that two of them may be chosen so that the distance between them is less than 1/2 unit apart.
   The proof is by dividing the triangle into 4 equilateral triangles so that 2 of the points lie inside one of them.

2. There is an element in the sequence 7, 77, 777, 7777, ... that is divisible by 2003.
   We shall prove that the first 2003 numbers actually have this property. Note that there 2002 possible nonzero remainders when divided by 2003. If none of these is divisible, then by the PHP at least one number from $1, \ldots, 2002$ occurs twice as a remainder. Say the i^{th} and j^{th} elements have this property. Subtracting them we see that $j - i$ digits are equal to 7 and the rest are 0. But this result has to be divisible by 2003. The cool thing is that $a_j - a_i = a_{j-i}10^i$ and hence we have a contradiction !