HW 2(to be tested on Jan 25)

1. Prove the well ordering principle for natural numbers using induction. As a corollary prove that any two natural numbers have a least common multiple, i.e., a number m which is a common multiple and which is $\leq$ any other common multiple.

2. (Modified Tower of Hanoi/Brahma - Problem 21 in Childs' book in the chapter on induction) The puzzle consists of n discs of decreasing diameters placed on a pole. There are two other poles placed in a row. The problem is to move the entire stack of discs to another pole by moving one disc at a time to an adjacent pole, except that no disc may be placed on top of a smaller disc. Find a formula for the least number of moves needed to move a stack of n discs from the leftmost pole to the right most pole. Prove the formula by induction.

3. Prove that

   (a) (Product principle) $|A_1 \times A_2 \ldots \times A_n| = |A_1||A_2| \ldots$.

   (b) (Sum principle) If $\cup_i A_i = C$ where A_i are pairwise disjoint, then $|C| = \sum_i |A_i|$.

   (c) (Quotient principle) If $\sim$ is an equivalence relation between elements of a set A and every equivalence class consists of k elements, then $|A| = kn$ where n is the number of equivalence classes. (In particular, $|A|$ is divisible by k and n.)

4. Prove that every permutation can be uniquely decomposed into disjoint unions of cycles.

5. Suppose $f(x) = \sum a_n x^n, g(x) = \sum b_n x^n, h(x) = \sum c_n x^n$ are two formal power series. Show that

   (a) $f(x)g(x) = g(x)f(x)$

   (b) $(f(x)g(x))(h(x)) = f(x)(g(x)h(x))$.

   (c) $(f(x) + g(x))h(x) = f(x)h(x) + g(x)h(x)$

   (d) the product rule holds, i.e., $h^{'}(x) = f^{'}(x)g(x) + g^{'}(x)f(x)$.

   (e) a power series has a multiplicative inverse if and only if its constant term is not zero.

   (f) $\left(\frac{1}{g}\right)^{'} = -\frac{g^{'}}{g^2}$ if g has a multiplicative inverse.