HW 6 (to be tested on Mar 15)

1. Suppose R is a commutative ring. For a given fixed $a \in R$, define the map $T_a : R[x] \to R$ as $T_a(p(x)) = p(a)$. This map is called the evaluation map. Prove that T_a is a homomorphism. What can go wrong if R is not commutative ?

2. For every field F there exists a field $\bar{F}$ which contains F as a subfield and is "algebraically closed", i.e., any polynomial $p(x) = a_0 + a_1 x + \ldots a_n x^n$ with coefficients $a_i \in \bar{F}$ has a root $u \in \bar{F}$. That is, $p(u) = 0$.

   (a) Prove that a polynomial $p(x)$ factors as $p(x) = a_n(x - u_1)(x - u_2)\ldots(x - u_n)$ where $u_i \in \bar{F}$.

   (b) Define the derivative of $p(x)$ as $p'(x) = a_1 + 2a_2 x + 3a_3 x^2 + \ldots$. Prove that $p(x)$ has a repeated root if and only if $p(x)$ and $p'(x)$ share a root.

   (c) Let $u_1, u_2, \ldots, u_{p^n} \in \bar{\mathbb{Z}}_p$ be the roots of $x^{p^n} = x$. Prove that this finite set is a subfield $\mathbb{F}_{p^n}$ of $\bar{\mathbb{Z}}_p$. Show also that the elements u_i are distinct and hence the finite field has order p^n.

3. Prove that $f(x), g(x) \in \mathbb{F}[x]$ where $\mathbb{F}$ is a field, then any common divisor of f, g divides any gcd of f, g produced by the Euclidean algorithm.

4. In $\mathbb{F}[x]$, every f factors uniquely (up to) units into a product of irreducible polynomials.