Notes for 10 Feb (Friday)

1 Compactness (cont'd)..

1. (HW 2) Imitate the proof of theorem 2.43 (in Rudin's book) to obtain the following result : If $\mathbb{R}^k = \cup_1^\infty F_n$ where each F_n is a closed subset of $\mathbb{R}^k$, then at least one F_n has a nonempty interior. (By the way, sets whose closure has empty interior are called nowhere dense.)
   Here is an equivalent statement (Why is this statement equivalent?) : If G_n are dense open subsets of $\mathbb{R}^k$ then $\cap_1^\infty G_n$ is not empty.

   Ans. There are two ways of solving this problem :
   First way : Suppose $x_1 \in G_1$. Then there is an $\tilde{r}_1$ such that $B_{\tilde{r}_1}(x_1) \subset G_1$ because G_1 is open. By shrinking $\tilde{r}_1$ to r_1 we may assume that the *closed* ball $\bar{B}_{r_1}(x_1) \subset G_1$. Since G_2 is dense, it means that every neighbourhood of every point intersects G_2. Therefore there exists $x_2 \in G_2 \cap B_{r_1}(x_1)$. Since $G_2 \cap B_{r_1}(x_1)$ is open, like before there exists $\bar{B}_{r_2}(x_2) \subset G_2 \cap B_{r_1}(x_1)$. Inductively, we may construct x_n and $\bar{B}_{r_n}(x_n)$ such that $\bar{B}_{r_n}(x_n) \subset B_{r_{n-1}}(x_{n-1}) \cap G_1 \cap G_2 \cap G_3 \ldots \cap G_n$. Thus $\bar{B}_{r_n}(x_n)$ are compact sets such that every finite intersection is not empty. Therefore their intersectin $\cap_n \bar{B}_{r_n}(x_n) \neq \phi$ which means there is some x in their intersection. I claim that $x \in G_n \; \forall \; n$. Indeed, $x \in \bar{B}_{r_n}(x_n) \cap G_n$.

   Second way : Firstly, let's prove that indeed the last statement is equivalent to the original one.
   Indeed, assume the last statement. Then if F_n are closed subsets of $\mathbb{R}^k$ such that $\cup F_n = \mathbb{R}^k$, then F_n^c are open subsets such that $\cap F_n^c = \phi$. But by the last statement, this means that at least one F_n^c is not dense. This means that there is a point $p \in \mathbb{R}^k$ such that p is not in F_n^c (so it is in F_n) and it is not a limit point of F_n^c. This means that there is a neighbourhood $B_r(p)$ such that no point of it is in F_n^c. This further means that $B_r \subset F_n$. Thus $p \in B_r(p) \subset F_n$. Therefore $p \in Int(F_n)$ which means that the interior is not empty.
   Assume the original statement. If G_n are dense open subsets of $\mathbb{R}^k$, and if $\cap G_n$ is empty, then we will derive a contradiction. Note that $F_n = G_n^c$ are closed subsets such that $\cup F_n = \mathbb{R}^k$. Therefore there is one F_n having nonempty interior. This means that there exists p and $B_r(p)$ such that $p \in B_r(p) \in F_n = G_n^c$. This means that p is NOT a limit point of G_n (and is certainly not in G_n). This means that G_n is not dense. Contradiction.

Secondly, let's prove the original statement itself - Suppose not, i.e., F_n are closed subsets whose union is $\mathbb{R}^k$ but every F_n has empty interior. Assume without loss of generality that the sets F_i are all distinct. (If not, simply keep only the distinct copies.)
Now choose a point $x_1 \in F_1$ and let V_1 be a neighbourhood of x_1. Assume without loss of generality that all the F_i intersect this neighbourhood in distinct points. (Otherwise just throw some of those F_i away because they do not matter for this argument.) Since the interior of F_1 is empty, there is a point x_2 in V_1 that is not in F_1. In fact, we can choose x_2 to belong to F_2. (If no such x_2 exists, then $F_2 \cap V_1 = F_1 \cap V_1$ and that is a problem by assumption.) Since F_1 is closed, there exists a neighbourhood V_2 around x_2 such that $\bar{V_2} \subset V_1$ and $\bar{V_2} \cap F_1 = \phi$. Now continue this way to produce points $x_n \in F_n$ and neighbourhoods V_n around x_n such that $\bar{V_n} \cap F_{n-1} = \phi$ and $\bar{V_n} \subset V_{n-1}$. Now the $\bar{V_n}$ are all compact sets (because they are closed and bounded). Hence by a theorem in the class, $\cap \bar{V_n}$ is not empty and contains a point p. The problem is that p is in some F_k but that is not possible because $F_k \cap \bar{V}_{k+1} = \phi$.

2. Let $K \subset \mathbb{R}$ consist of 0 and the numbers $\frac{1}{n}$. Then prove that K is compact using the definition.
Ans. Suppose U_α is an open cover of K. Since $0 \in U_\beta$ for some β, and the U_β is open, of course $(-\epsilon, \epsilon) \subset U_\beta$ for some small ϵ. This means that along with 0 all but finitely many $\frac{1}{n}$ are in U_β. The rest of the finitely many $\frac{1}{n}$ that are not in U_β are of course in finitely many open sets $U_{\alpha_1}, U_{\alpha_2}, \ldots U_{\alpha_k}$. This proves it.

2 Connectedness

A set E is said to be connected iff E is NOT equal to $(U_1 \cap E) \cup (U_2 \cap E)$ where U_1, U_2 are open subsets of X and $U_1 \cap E \cap U_2 \cap E = \phi$. In simpler terms, E is connected if it is not a collection of two disjoint relatively open sets.
$E \subset \mathbb{R}$ is connected if and only if for every $x < y$, all z such that $x < z < y$ are in E.

Here is a problem (that can be solved using connectedness but is easier to solve otherwise) :
(HW 2) Prove that every open set in $\mathbb{R}^1$ is the union of an at most countable collection of disjoint open intervals.
Ans. Take all the rationals q_n in the open set E of $\mathbb{R}$. Because E is open, $(q_n - \epsilon_n, q_n + \epsilon_n) \subset E$ for some $\epsilon_n > 0$. Let $M_n = \sup_{t \in E}(q_n - \epsilon, t) \subset E$ and $m_n = \inf_{s \in E}(s, q_n + \epsilon) \subset E$. I claim that $(m_n, M_n) \subset E$. Indeed, if $x \in (m_n, M_n)$ and $x > q_n$ then by definition of supremum, there exists $y \in E$ such that $y > x$ and $x \in (q_n - \epsilon, y) \subset E$. Likewise if $x < q_n$. Also, $E = \cup_n (m_n, M_n)$. Indeed, if $x \in E$ then $(x - \epsilon, x + \epsilon) \subset E$. By density of rationals, there is a rational $q_n \in (x - \epsilon, x + \epsilon) \subset E$. Therefore, $x \in (m_n, M_n)$. I also claim that if $(m_n, M_n) \cap (m_j, M_j) \neq \phi$ then $(m_n, M_n) = (m_j, M_j)$. Indeed if not, then either $m_n < m_j$ or $m_n > m_j$ or $M_n < M_j$ or $M_n > M_j$. Without loss of generality $M_n < M_j$ (the arguments are similar in the other cases). Then, note that $(q_n - \epsilon_n, M_j) \subset E$. This is a contradiction to the assumption that M_n is the supremum of all t such that $(q_n - \epsilon_n, t) \subset E$.

3 Sequences

A sequence is simply an infinite (countable) list of elements of X, $x_1, x_2, \ldots$. It converges to x if for every $\epsilon > 0$ there exists $N > 0$ such that $d(x_n, x) < \epsilon \ \forall\ n > N$.
A subsequence is a subcollection of the infinite list $x_{n_1}, x_{n_2}, \ldots$ where $n_1 < n_2 < n_3$.....
A subsequential limit is $\lim_{k\to\infty} x_{n_k}$. A sequence converges to x if and only if all its convergent subsequences converge to x.

Problem : A point p is a limit point of a sequence $x_1, x_2, \ldots$ if and only if there is a subsequence $x_{n_k} \to p$ where all but finitely many terms of the subsequence are different from p.
Ans) Indeed, suppose there is such a subsequence. This means that given any $\epsilon > 0$ there exists an N such that $k > N \to d(x_{n_k}, p) < \epsilon$. This means that given any neighbourhood $B_\epsilon(p)$, there exists a point $x_{n_k} \neq p$ in $B_\epsilon(p)$. Therefore it is a limit point. Conversely, suppose p is a limit point of the sequence. This means that for every neighbourhood $B_{1/k}(p)$ there exists a point $x_{n_k} \neq p \in B_{1/k}(p)$. In addition, we may assume that $n_k > n_{k-1} > \ldots$ because if inductively, this is true for $k-1$ then since there are only finitely many $l < n_{k-1}$ there exists some $l = n_k > n_{k-1}$. (Note that we proved that for a limit point of a set, every neighbourhood actually consist of infinitely many points from the set.) Thus for every $\epsilon > 0$, choosing an integer $N > 0$ such that $\frac{1}{N} < \epsilon$ (by the Archimedian property) the points x_{n_k} for all $k > N$ lie in $B_{1/N}(p) \subset B_\epsilon(p)$. Therefore by definition $x_{n_k} \to p$.

A bounded sequence in $\mathbb{R}^n$ has a convergent subsequence. This is the content of the Bolzano-Weierstrass theorem.

A Cauchy sequence x_n is one where eventually all the terms are very close to one another, i.e., given any $\epsilon > 0$ there is an $N_\epsilon > 0$ such that $n, m > N_\epsilon$ implies that $d(x_n, x_m) < \epsilon$. Please note that in $\mathbb{R}^n$ every Cauchy sequence converges. So in a problem if I ask you to prove that a sequence in $\mathbb{R}^n$ (does not work for other metric spaces) converges, one possible strategy is to prove that it is a Cauchy sequence.

Another useful point : To prove that a Cauchy sequence x_n converges to x, it is enough to find one subsequence that converges to x.

Fact : If a set K is compact then every sequence has a convergent subsequence whose limit is in K.

For example, If you take the following sequences, what are all possible subsequences and the subsequential limits?

1. $1, -1, 1, -1, 1, \ldots$: Either you have infinitely many 1 and -1 in your subsequence, or finitely many 1 or finitely many -1. (So the subsequential limits are only 1 and -1. The sequence itself does not converge.)

2. $a_n = 1, -2, 3, 0, 1, -2, 3, 0, \ldots$: In any subsequence, either all but finitely many are 1 xor -2 xor 3 xor 0, or not. In the former case, the subsequence converges to $1, -2, 3$ xor 0. In the latter case, the subsequence fails to converge.

In the above examples, the limit of the sequence fails to exist but that of some subsequences does exist. For *any* real sequence, there are two kinds of subsequential limits that *always* exist (if you allow $\pm\infty$ as valid limits). These are the $\limsup$ and $\liminf$. The lim sup is the supremum of all subsequential limits. In fact, it is the maximum of all subsequential limits. (Likewise lim inf.) The lim sup and lim inf of the above examples are:

1. lim sup is 1 and lim inf is -1.
2. lim sup is 3 and lim inf is -2.

The point is that if $x > \limsup a_n$ then there is an N such that $x > a_n \forall n > N$. Also, there is a mechanical way to calculate lim sup and lim inf. You do not need to find out all subsequential limits. The formulae are :
$\limsup a_n = \lim_{N\to\infty} \sup_{n\geq N} a_n$.
$\liminf a_n = \lim_{N\to\infty} \inf_{n\geq N} a_n$.

Why should you care about lim sup and lim inf ? How will you use them in problems (that are not as straightforward as "Tell me the lim sup of this sequence")? The main points are that :

1. A sequence $a_n \to a$ if and only if $\limsup a_n = \liminf a_n = a$.
2. Also, $\liminf a_n \leq \limsup a_n$ always exist for any real sequence (if you allow $\pm\infty$).
3. If $a_n \leq b_n$ then $\limsup a_n \leq \limsup b_n$ and $\liminf a_n \leq \liminf b_n$.

For instance, (although series are not there in the syllabus) we used these facts to prove that $\lim_{n\to\infty}(1 + 1/n)^n = e$.

Here are some problems :

1. If $s_1 = \sqrt{2}$, and $s_{n+1} = \sqrt{2 + \sqrt{s_n}}$ then prove that s_n converges and that $s_n < 2$.
   Ans. $s_1 < 2$. Inductively assume that $s_n < 2$. Then $s_{n+1} < \sqrt{2 + \sqrt{2}} < \sqrt{4} < 2$. Thus the sequence is bounded. I also claim that $0 < s_n < s_{n+1}$. For $n = 1$, $\sqrt{2+\sqrt{2}} > \sqrt{2}$. Inductively assume that $s_n > s_{n-1}$, i.e. $\sqrt{s_n} > \sqrt{s_{n-1}}$. (This is true because if not, just square on both sides to get a contradiction.). Indeed, $s_{n+1}^2 - s_n^2 > \sqrt{s_n} - \sqrt{s_{n-1}} > 0$ inductively. Thus $s_{n+1} - s_n > 0$ (because $s_{n+1} + s_n > 0$). Since a monotone bounded sequence converges we are done.
2. Fix $\alpha > 1$. Take $x_1 > \sqrt{\alpha}$. Define $x_{n+1} = \frac{\alpha + x_n}{1 + x_n} = x_n + \frac{\alpha - x_n^2}{1 + x_n}$. Prove that
   (a) $x_1 > x_3 > \ldots$.
   (b) $x_2 < x_4 < \ldots$.
   (c) $\lim x_n = \sqrt{\alpha}$.

Firstly, inductively all $x_n > 0$. (Indeed this is true for $n = 1$. Assuming truth for n, by the first equality defining x_{n+1} it is easily seen to be true for $n+1$.) We prove a) and b) simultaneously. Indeed, note that

$$\begin{aligned} x_k - x_{k-2} &= \frac{\alpha + x_{k-1}}{1 + x_{k-1}} - \frac{\alpha + x_{k-3}}{1 + x_{k-3}} \\ &= \frac{(\alpha - 1)(x_{k-3} - x_{k-1})}{(1 + x_{k-1})(1 + x_{k-3})} \end{aligned} \tag{1}$$

Claim : $x_{2n-1} > x_{2n+1}$ and $x_{2n} < x_{2n+2}$.
For $n = 1$: $x_3 - x_1 = \frac{\alpha + x_2}{1 + x_2} - x_1 = \frac{\alpha + x_2 - x_1 - x_1 x_2}{1 + x_2}$. Indeed,this simplifies to

$$\begin{aligned} (1 + x_2)(x_3 - x_1) &= \alpha - x_1 + \frac{\alpha + x_1}{1 + x_1}(1 - x_1) \\ &= \frac{2(\alpha - x_1^2)}{1 + x_1} < 0. \end{aligned} \tag{2}$$

Putting $k = 4$ in 1 we see that $x_4 - x_2 > 0$ because $\alpha > 1$ and $x_1 > x_3$.
Assuming truth for n, we will prove for $n + 1$. Indeed, putting $k = 2n + 3$ in equation 1 we see that since $x_{k-3} - x_{k-1} = x_{2n} - x_{2n+2} < 0$ and $\alpha - 1 > 0$ we get $x_{2n+3} - x_{2n+1} < 0$. Also, putting $k = 2n+4$ in 1, since $x_{k-3} - x_{k-1} = x_{2n+1} - x_{2n+3} > 0$ we get $x_{2n+4} > x_{2n+2}$ which is what we wanted.

We also claim that $x_{2n} < \sqrt{\alpha}$ and $x_{2n+1} > \sqrt{\alpha}$. Indeed, note that $x_{k+1} - \sqrt{\alpha} = \frac{\alpha + x_k - \sqrt{\alpha} - \sqrt{\alpha} x_k}{1 + x_k} = \frac{(-x_k + \sqrt{\alpha})(-1 + \sqrt{\alpha})}{1 + x_k}$. Therefore the claim is true for $n = 1$ and inductively for all n.

Thus the even subsequence is increasing and bounded. The odd subsequence is decreasing and bounded. So both of them converge to something(s). Let's say $\lim x_{2n+1} = x$ and $\lim x_{2n} = y$. Now

$$\begin{aligned} x_{2n+1} &= \frac{\alpha + x_{2n}}{1 + x_{2n}} \\ x_{2n} &= \frac{\alpha + x_{2n-1}}{1 + x_{2n-1}}. \end{aligned}$$

Taking the limit as $n \to \infty$ on both sides of the equations above,

$$\begin{aligned} x &= \frac{\alpha + y}{1 + y} \\ y &= \frac{\alpha + x}{1 + x} \end{aligned}$$

Solving for x and y we get $x = y = \sqrt{\alpha}$.

We shall calculate lim sup and lim inf and prove that they are equal to $\sqrt{\alpha}$. Indeed,

$$\limsup x_n = \lim_{N \to \infty} \sup_{n \geq N} x_n = \lim_{N \to \infty} b_N$$

where $b_N = x_N$ if N is odd and $b_N = x_{N+1}$ if N is even. In either case, the sequence b_N is $x_1, x_3, x_3, x_5, x_5 \ldots$ which converges to $\sqrt{\alpha}$ because it is monotonically increasing and bounded above and has a subsequence that converges to $\sqrt{\alpha}$. Likewise, we can calculate the lim inf to be $\sqrt{\alpha}$.