Notes for 10 Mar (Friday)

1 Recap

1. Proved that if f has only finitely many discontinuities and α is continuous there, then f is RS.
2. Proved that if α is continuous and f monotonic, then also f is RS.
3. If f is RS and g continuous, then $g \circ f$ is RS.
4. Proved the usual properties of integrals.

2 Properties of RS functions

Here is another property.

Lemma 2.1. *If f and g are RS integrable then*

1. *So is fg.*
2. *$|f|$ is so and $|\int f d\alpha| \leq \int |f| d\alpha$.*

Proof. 1. The elegant idea is $fg = \frac{(f+g)^2-(f-g)^2}{4}$. Since f, g are RS integrable so are $f+g$ and $f-g$. Since x^2 is a continuous function, $(f+g)^2$ and $(f-g)^2$ are RS integrable. Their difference (divided by 4) is also RS integrable.

2. $\phi(t) = |t|$ is continuous and hence $|f|$ is RS integrable. Now let $c = \pm 1$ be chosen so that $c\int f d\alpha \geq 0$. Therefore $|\int f d\alpha| = c\int f d\alpha = \int cf d\alpha \leq \int |f| d\alpha$ because $cf \leq |f|$.

□

Let $s(x) = 0$ when $x \leq 0$ and $s(x) = 1$ when $x > 0$. We naively expect $s^{'}(x) = \delta(x)$ (although the Dirac delta is not a function).

Theorem 1. *If $a < t < b$, f is bounded on $[a, b]$, $\alpha(x) = s(x-t)$, and f is continuous at t then $\int_a^b f d\alpha = f(t)$.*

Proof. Suppose P is any partition of $[a, b]$ not containing t such that $x_{i-1} \leq t \leq x_i$. Then $U(P, f, \alpha) = M_i$ and $L(P, f, \alpha) = m_i$. Since f is continuous at t, there is a $\delta > 0$ such that $|f(x) - f(t)| < \epsilon/2$ whenever $|x-p| < \delta$. Choosing the partition such that $x_i - t < \delta$ we see that $M_i \leq f(t) + \epsilon/2$ and $m_i \geq f(t) - \epsilon/2$. Thus $U(P) - L(P) < \epsilon$. Thus f is RS integrable. Moreover, $|\int f d\alpha - \sum f(v_i)\Delta\alpha_i| < \epsilon$. Thus $|\int f d\alpha - f(t)| < \epsilon$ for every ϵ. Hence we are done. □

In fact, something stronger is true.

Theorem 2. *Suppose $c_n \geq 0$, $\sum c_n$ converges, t_n is a sequence of distinct points in $[a, b]$ and $\alpha(x) = \sum_{n=1}^{\infty} c_n s(x - t_n)$. Let f be continuous. Then $\int_a^b f d\alpha = \sum c_n f(t_n)$.*

Proof. Firstly, the comparison test (along with the boundedness of f because of the extreme value theorem) shows that the series on the right hand side converges. Actually this also shows that $\alpha(x)$ is well-defined. It is easy to see that α is increasing.
Secondly, suppose N is so large that $\sum_{n=N+1}^{\infty} c_n < \epsilon$. Then let $\alpha_1(x) = \sum_{n=1}^{N} c_n s_n(x - t_n)$ and $\alpha_2(x) = \sum_{N+1}^{\infty} c_n s_n(x - t_n)$. Definitely α_1, α_2 are well-defined. They are also increasing. Because f is continuous, f is RS integrable w.r.t α, α_1, and α_2. Moreover, $\int_a^b f d\alpha = \int_a^b f d\alpha_1 + \int_a^b f d\alpha_2$. Thus

$$\int_a^b f d\alpha = \sum_{n=1}^{N} \int c_n f ds_n(t - x_n) + \int_a^b f d\alpha_2 = \sum_{n=1}^{N} c_n f(t_n) + K(\alpha_2(b) - \alpha_2(a)). \quad (1)$$

Let $N \to \infty$ to see that $| \int_a^b f d\alpha - \sum c_n f(t_n)| \leq K(\alpha_2(b) - \alpha_2(a))$. But $0 < \alpha_2(b) - \alpha_2(a) < \epsilon$. Thus tending $\epsilon \to 0$ we get the result. □

We can now connect the RS integral to the usual Riemann integral.

Theorem 3. *Suppose α is increasing, and α' is Riemann integrable. Suppose f is a bounded real function. Then f is RS w.r.t α if and only if $f\alpha'$ is Riemann integrable. Moreover, $\int f d\alpha = \int f\alpha' dx$.*

Proof. Since α' is integrable, there exists a partition P so that $\sum |\alpha'(s_i) - \alpha'(t_i)|\Delta x_i < \epsilon$ for any choices s_i, t_i in $[x_{i-1}, x_i]$.
By the MVT $\Delta\alpha_i = \alpha'(c_i)\Delta x_i$. Put $M = \sup |f(x)|$. So $\sum f(s_i)\Delta\alpha_i = \sum f(s_i)\alpha'(t_i)\Delta x_i$. Thus $| \sum f(s_i)\Delta\alpha_i - \sum f(s_i)\alpha'(s_i)\Delta x_i \leq M\epsilon$. Therefore $\sum f(s_i)\Delta\alpha_i \leq U(P, f\alpha') + M\epsilon$. Hence $U(P, f, \alpha) \leq U(P, f\alpha', x) + M\epsilon$. Likewise, $U(P, f\alpha') \leq U(P, f, \alpha) + M\epsilon$. This leads us to conclude that $| \int_a^{\bar{b}} f d\alpha - \int_a^{\bar{b}} f\alpha' dx| \leq M_\epsilon$. A similar argument for the lower integrals shows the result. □

There are (not-so-easy to describe) counterexamples that show that α just being differentiable everywhere is not good enough for $\int f d\alpha = \int f\alpha' dx$.

The next order of business is change-of-variable of integration, i.e., the ability to evaluate integrals by substitution.

Theorem 4. *Suppose ϕ is strictly increasing and continuous that maps $[A, B]$ onto $[a, b]$. Suppose α is increasing on $[a, b]$ and f is RS w.r.t α. Let $\beta(y) = \alpha(\phi(y))$ and $g(y) = f(\phi(y))$. Then g is RS w.r.t β and $\int_A^B g d\beta = \int_a^b f d\alpha$.*

Proof. Given a partition $P = \{x_0 = a \leq x_1 \leq \ldots x_n = b\}$ we can get a partition $Q = \{y_0 = A, y_1 = \phi^{-1}(x_1) \ldots$. So if we choose a partition P such that $\sum(M_i - m_i)\Delta\alpha_i < \epsilon$, then the partition Q of $[A, B]$ is such that $\sup_{y \in [y_{i-1}, y_i]} g(y) = \sup_{y \in [y_{i-1}, y_i]} f(\phi(y)) = \sup_{x \in [x_{i-1}, x_i]} f(x) = M_i$ and likewise for m_i. Moreover, $\Delta\beta_i = \beta(y_i) - \beta(y_{i-1}) = \alpha(x_i) -$

$\alpha(x_{i-1}) = \Delta\alpha_i$. Thus $U(Q) - L(Q) < \epsilon$. Therefore $\int g d\beta$ exists. Also since $U(Q) = U(P)$, $U(Q) \leq \int_a^b f d\alpha$ and so on. Likewise for the lower integral. Thus the integrals are equal. □

3 Fundamental theorems of calculus

Here is the theorem that states that if you differentiate an integral you get the function back.

Theorem 5. *Let f be Riemann integrable on $[a, b]$. Define $F : [a, b] \to \mathbb{R}$ as $F(x) = \int_a^x f(t)dt$. Then F is continuous. Moreover, if f is continuous at $x_0 \in (a, b)$ then F is differentiable there and $F'(x_0) = f(x_0)$.*

Proof. Firstly, $F(x)$ is well-defined because $F(x) = \int_a^b f(t)g(t)dt$ where $g(t) = 1 - s(t-x)$ and f, g are Riemann integrable. By the properties of Riemann integrals, if $y \leq x$, then $|F(x) - F(y)| = |\displaystyle\int_x^y f(t)dt| \leq M|y - x| < \epsilon$ if $|y - x| < \frac{\epsilon}{M}$. So F is (uniformly) continuous. Moreover, if f is continuous at x_0, then

$$\begin{aligned} \left|\frac{F(x) - F(x_0)}{x - x_0} - f(x_0)\right| &= \left|\frac{\int_{x_0}^x f(t)dt}{x - x_0} - f(x_0)\right| \\ &= \left|\frac{\int_{x_0}^x (f(t) - f(x_0))dt}{x - x_0}\right| \leq \epsilon, \end{aligned}$$

if $|x - x_0| < \delta$. Thus we are done. □