Notes for 12th Jan (Thursday)

1 A longish recap

1. We "defined" sets as "unordered collections" of "objects" (just like in high school). All that you learnt in high school set theory is fine for all practical purposes. It is just that you need to be careful (not everything is a set. You can come up with new sets only from old ones from "usual operations like subsets, power set, union, intersection, etc). But you do not need to worry about set theory at all. (I never saw the ZFC axioms in detail until I started teaching you chaps! I do research on a daily basis!)

2. We constructed natural numbers and stated Peano's axioms for them. Then we defined addition, ordering, and multiplication for them. Essentially, every thing you learnt in second standard is valid !

3. We constructed integers and rational numbers from natural numbers. We stated their usual properties. In other words, we can manipulate fractions to our heart's content. So fifth standard maths is fine. Moreover, we are assumed to know about the absolute value function and its properties (like the triangle inequality). This is a part of 11th and 12 th standard maths (and first year at IISc).

4. The real deal came when we noticed that $\sqrt{2}$ does not make sense as a rational number. So we constructed real numbers out of rational numbers in a particular manner (and we will continue to talk about it in this class). But at the end of the day, real numbers obey addition, multiplication, etc (just like rationals) except that every subset that is bounded above has a least upper bound. (You would not have used this property in school. Maybe in calculus in 11th, 12th, and the first year at IISc you would have used the notion of limits. But even then, what exactly distinguishes rationals from reals may not have been taught. It is the least upper bound property.)

5. Reals are in plain English - "Sequences of rational numbers that become very close to one another eventually. Two sequences talk about the same real number if these two sequences get very close to each other eventually."
   We defined addition of real numbers and made sure that the definition made sense. (Unlike the chap who says "The nation wants to know".)

6. Today we hope to define multiplication, and ordering of real numbers. Moreover, we hope to sketch a proof that our construction indeed satisfies the least upper bound property.

2 Real numbers (cont'd)...

Multiplication : If $x = [a_i]$ and $y = [b_i]$ then xy is defined to be the Cauchy sequence $[a_i b_i]$. (Why is this Cauchy ?)
Negation : If $x = [a_i]$ then $-x = [-a_i]$.
Reciprocal : If $x = [a_i] \neq 0$ then we may assume without loss of generality that $a_i \neq 0$ for any i (why?). Define $x^{-1} = [\frac{1}{a_i}]$.
Ordering : $x = [a_i]$ is said to strictly positive if there exists a natural N so that $a_i > 0 \; \forall \; i > N$. Also, $x > y$ if $x = y + z$ for some strictly positive z.

You need to verify that indeed the above definitions make sense (that is the quantities are well-defined and that they satisfy the axioms of an ordered field). I leave this as a masochistic exercise to you. We can further define the absolute value function $|x|$ from reals to positive reals. Also, rationals sit inside the reals as $p/q \to (p/q, p/q, p/q, \ldots)$.

Let's verify the sup property. But to do this we need to prove two lemmas.
' The first one is called "Density of rationals" (In plain English, between any two reals there is a rational).

Lemma 2.1. *If $x < y \in \mathbb{R}$ then there exists a rational p/q such that $x < p/q < y$.*

The second one is called the Archimedian property of reals.

Lemma 2.2. *If x and y are two reals, and $x > 0$ then there exists a positive integer m such that $mx > y$.*

Now we return to the proof of the sup property. Let S be a non-empty subset of reals bounded above by a real number M, i.e. $s \leq M \; \forall \; s \in S$ and $x_0 \in S$. By the Archimedian property given an integer $n \geq 1$ there exists a K such that $\frac{K}{n} \geq M$. Likewise, $\exists L | \frac{L}{n} \leq x_0$. Thus there exists a unique integer m_n such that $\frac{m_n}{n}$ is an upper bound but $\frac{m_n - 1}{n}$ is not. The sequence $\frac{m_n}{n}$ may be verified to be a Cauchy sequence of rationals and is also the supremum of S. (See Terence Tao's book.)

Now we prove that nth roots of positive reals exist, i.e.,

Lemma 2.3. *For every positive real $x > 0$ and every integer $n > 0$ there exists a unique positive real y such that $y^n = x$.*

Proof. Uniqueness is clear. (Why?) The problem is existence.

How does one solve this problem on a computer ? One "guesses" at the answer y_1 such that $y_1^n < x$. Then one improves this guess by adding something to y_1 to get y_2 and so on. Then one hopes that this sequence converges to the correct answer. (One way to do the iteration is by the Newton-Raphson scheme.)
One tries something similar here except that we shall use the least upper bound property. So let E be the set of all positive reals t such that $t^n < x$. This set is of course non-empty. The reason is that if $x > p/q$ for some rational (it exists by the Archimedian property) then $(1/q)^n = 1/q^n < p/q^n < p/q < x$. If $t = 1 + x$ then $t^n > 1 + x^n > x$. (Why?) So this set is bounded above and hence the least upper bound y exists. We claim that $y^n = x$.

Indeed, Suppose

1. $y^n < x$: Then if we come up with $z = y + \epsilon$ satisfying $z \in E$ then it contradicts the fact that y is an upper bound of E. To do so, if we make sure that $(y+\epsilon)^n - y^n < x - y^n$ we should be in good shape, i.e.,

$$(y+\epsilon)^n - y^n = \epsilon(y^{n-1} + \ldots + (y+\epsilon)^{n-1}) < n2^{n-1}y^{n-1}\epsilon \tag{1}$$

   if ϵ is chosen to be less than y. Thus if we choose ϵ to be less than y and less than $\frac{x-y^n}{n2^{n-1}y^{n-1}}$ we will be done.

2. $y^n > x$: Then $z = y - \epsilon$ for an appropriate ϵ (like before) still satisfies $z^n > x$ contradicting the fact that y is the least upper bound of E.

Therefore, $y^n = x$. □