Notes for 15 Mar (Wednesday)

1 Recap

1. We proved that if f, g are RS then so are fg and $|f|$.
2. We proved the intuitively clear statement that if f is continuous, $\alpha = \sum c_n s(x - t_n)$ where $\sum c_n$ converges ($c_n \geq 0$) then $\int f d\alpha = \sum c_n f(t_n)$.
3. We connected RS to Riemann integrals in the case when $\alpha^{'}$ is Riemann integrable.
4. We proved the change of variables theorem.
5. We proved that if f is continuous at a point then the derivative of the integral of f equals f at that point.

2 Fundamental theorems of calculus

In the second one if you prove that antiderivatives can be used to calculate integrals.

Theorem 1. *If f is Riemann integrable and if there is a differentiable F such that $F^{'} = f$ then $\int_a^b f(x)dx = F(b) - F(a)$.*

Proof. Choose a partition P such that $U(P, f) - L(P, f) < \epsilon$. By the MVT, $F(x_i) - F(x_{i-1}) = f(t_i)\Delta x_i$. Note that $|\sum f(t_i)\Delta x_i - \displaystyle\int_a^b f(x)dx| < \epsilon$ and hence $F(b) - F(a) = \int_a^b f(x)dx$. □

Finally, we have the integration-by-parts formula for differentiable functions F and G such that $F^{'} = f$ and $G^{'} = g$ are Riemann integrable. Actually, we also have the following integration-by-parts formula for the RS integral.

Theorem 2. *If f and α are monotonically increasing such that f is RS w.r.t α then $\int_a^b f d\alpha = f(b)\alpha(b) - f(a)\alpha(a) - \int_a^b \alpha df$.*

Proof. Note that $U(P, f, \alpha) - L(P, f, \alpha) = \sum (M_i - m_i)\Delta\alpha_i = \Delta f_i \Delta\alpha_i$. Since this is symmetric, and f is RS w.r.t α, so is α RS w.r.t f. Now $U(P, f, \alpha) + L(P, \alpha, f) = \sum f(x_i)(\alpha(x_i) - \alpha(x_{i-1})) + \alpha(x_{i-1})(f(x_i) - f(x_{i-1})) = f(b)\alpha(b) - f(a)\alpha(a)$. Since there exists a partition P (after taking a common refinement if necessary) such that $\int f d\alpha < U(P, f, \alpha) < \int f d\alpha + \epsilon$ and $\int \alpha df - \epsilon < L(P, \alpha, f) < \int \alpha df$, we see that since ϵ is arbitrary, we are done. □

3 Integration of vector-valued functions

If $\vec{f}:[a,b]\to\mathbb{R}^n$ is a vector-valued bounded function, and $\alpha:[a,b]\to\mathbb{R}$ is montonically increasing, then we define $\int_a^b \vec{f}d\alpha = (\int_a^b f_1 d\alpha,\ldots)$. The usual properties of integrals are valid (by just applying these results to each of the components). For instance, if $\vec{F}' = \vec{f}$ then $\int_a^b \vec{f}dt = \vec{F}(b)-\vec{F}(a)$. However, there is one theorem whose proof is slightly non-trivial.

Theorem 3. *If $\vec{f}$ maps $[a,b]$ into $\mathbb{R}^n$ and if $\vec{f}$ is RS integrable, then $\|\vec{f}\|$ is RS integrable and $\|\int_a^b \vec{f}d\alpha\| \leq \int_a^b \|\vec{f}\|d\alpha$.*

Proof. Firstly note that f_i^2 are RS integrable because x^2 is continuous and f_i are RS. Then note that $\|\vec{f}\|^2 = \sum f_i^2$ is RS. Now I claim that the square root function is continuous and hence $\|\vec{f}\|$ is RS. Indeed the sequare root function $\sqrt{x}:[a,b]\to[A,B]$ is the inverse of the square function $x^2:[A,B]\to[a,b]$ which is a continuous bijection from a compact set to another set. Therefore its inverse is continuous.

Note that $|\int_a^b \vec{f}.\vec{g}d\alpha| \leq \int_a^b \|\vec{f}\|\|\vec{g}\|d\alpha$. Taking $\vec{g}$ to be the constant function $\vec{g} = \int_a^b \vec{f}d\alpha$ we are done. □

4 Rectifiable curves

A curve γ in $\mathbb{R}^k$ is simply a continuous map of an interval $[a,b]$ to $\mathbb{R}^k$. If γ is 1-1 it is called an arc. If $\gamma(a)=\gamma(b)$ it is called a closed curve. (If $\gamma(x)\neq\gamma(y)$ except for the endpoints, then it is called a simple closed curve.) Note that a curve is a map. So $\gamma_1:[0,1]\to\mathbb{R}^2$ defined by $\gamma_1(t)=(t,t)$ and $\gamma_2;[0,2]\to\mathbb{R}^2$ defined by $\gamma_2(t)=(\frac{t}{2},\frac{t}{2})$ are different curves having the same range.
Our aim is to define the length of a curve. Naively we might want to say that the length is $\int\|\gamma'\|dt$. But what is γ is not differentiable everywhere ? So we need a more general definition for the length of a curve.
Given a partition P of $[a,b]$ we define the number $\Lambda(P,\gamma) = \sum_{i=1}^{n}\|\Delta_i\gamma\|$ where $\Delta_i\gamma = \gamma(x_i)-\gamma(x_{i-1})$. The length of γ is defined as $L(\gamma)=\sup_P \Lambda(P,\gamma)$. This supremum exists in the extended real number system. If $L(\gamma)<\infty$ then the curve is said to be rectifiable. In some cases, this is given by a Riemann integral. In particular, this is the case if γ has continuous derivatives. (These functions are called C^1 sometimes.)

Theorem 4. *If γ' is continuous on $[a,b]$ then γ is rectifiable and $L(\gamma) = \int_a^b \|\gamma'(t)\|dt$.*

Proof. Let M be such that $\|\gamma'\|\leq M$ on $[a,b]$. M exists because γ' is continuous. Thus by the MVT $\|\Delta\gamma_i\|\leq M(b-a)$. This means that $L(\gamma)<\infty$. Thus the curve is rectifiable.

Each $\gamma_i^{'}$ is continuous on a compact set $[a, b]$ and is thus uniformly continuous. So given a $\delta > 0$ there exists an N_i such that $|t - s| < \frac{b-a}{N_i}$ implies that $|(\gamma^{'})^2(t_i) - (\gamma^{'})^2(s_i)| < \delta$. Choose $N > max(N_i)$ for all i. Assume that the partition P is $x_0 = a \leq x_1 = a + \frac{b-a}{N} \ldots$. Also, choose δ so that $\sqrt{x} - \sqrt{x_0} < \epsilon$ whenever $|x - x_0| < \delta k$.
By the usual MVT, for each γ_i there exists a $c_{i,j}$ in $[x_{j-i}, x_j]$ s.t. $\Delta\gamma_i = \gamma^{'}(c_{i,j})\Delta t_i$. Therefore $\Lambda(P, \gamma) = \displaystyle\sum_j \|\Delta_j \gamma\| = \sum_j \sqrt{\sum_i \gamma^{'}(c_{i,j})^2} \frac{b-a}{N}$. Let $v_j = \frac{x_{j-1}+x_j}{2}$. Thus $|\Lambda(P, \gamma) - \displaystyle\sum_j \|\gamma^{'}(v_j)\| \frac{b-a}{N}| < \epsilon(b-a)$. As $N \to \infty$, we see that $\sum \|\gamma^{'}(v_j)\| \frac{b-a}{N} \to \displaystyle\int_a^b \|\gamma^{'}\| dt$. Thus make $N \to \infty$ and then $\epsilon \to 0$ to be done. $\square$

Here is a famous example of a non-rectifiable curve. I will not discuss it rigorously though. This is the Koch snowflake curve :
Take an equilateral triangle of side 1. Trisect each of its sides, remove the middle pieces, and instead replace it with two sides of an equilateral triangle of side 1/3. Continue this process. So each time, the perimeter P_n of the resulting curve C_n is 4/3 of that of C_{n-1}. Thus the perimeter runs off to infinity. The "limit" (in some appropriate sense) of C_n is the so-called Koch snowflake curve. It is clearly not rectifiable. For fun, let's calculate the area of the Koch snowflake curve. Note that to A_{n-1} we add the area of an equilateral triangle for every line segment. Suppose the number of line segments after $n - 1$ iterations is k_{n-1}. Then $k_n = 4k_{n-1}$ with $k_0 = 3$. Thus $k_n = 4^n 3$. The number of new triangles after n iterations is K_{n-1}. The length of every segment after n iterations is $\frac{1}{3^n}$. Thus $A_n = A_{n-1} + 4^{n-1} \times 3 \times \frac{3}{4}\frac{1}{9^n}$. Therefore $A = \frac{\sqrt{3}}{4}\frac{8}{5}$. So the area is finite but the perimeter is infinite!