Notes for 18th Jan (Wednesday)

1 Recap

1. We finished our discussion of real numbers. We proved that our construction (using Cauchy sequences of rationals) satisfies the least upper bound property. We discussed decimal expansions of reals. (So all of our high school maths is fine.) We also proved that the cardinality of reals is the same as the power set of natural numbers. (Cantor's diagonalisation.)

2. We defined Complex numbers and stated their properties.

3. We studied the fun example of Pythagorean triples.

2 Euclidean space (Cont'd..)

As discussed the last time, the study of Pythagorean triples almost forces us to study plane geometry. Likewise, any analytically minded human being would ask "What about five integers such that the sum of their squares is a perfect square?" and so on. This will force us to study geometry in higher dimensions. There are of course even clearer reasons to study geometry in higher dimensions (least squares fitting, computer graphics, building a robotic arm, statistical mechanics, life, etc).

Thus we make $\mathbb{R}^n$ into an object where things can be added.
Definition : Let $\vec{a} = (a_1, \ldots, a_n)$ and $\vec{b} = (b_1, \ldots, b_n)$. Then $\alpha\vec{a}$ is defined as $(\alpha a_1, \ldots, \alpha a_n)$ and $\vec{a} + \vec{b} = (a_1 + b_1, \ldots, a_n + b_n)$. (It is a "vector space".)

Since we want to do geometry, we need to define a notion of distance. Indeed, we define the length ("the norm") of a vector $\vec{x}$ via the following equation.

$$\|\vec{x}\|^2 = x_1^2 + x_2^2 + \ldots + x_n^2 \tag{1}$$

This length thing satisfies the following properties :

1. $\|\vec{x}\|^2 \geq 0$ with equality holding if and only if $\vec{x} = 0$.

2. $\|\lambda\vec{x}\|^2 = \lambda^2\|\vec{x}\|^2$.

3. (Triangle inequality) $|\|\vec{x}\| - \|\vec{y}\|| \leq \|\vec{x} + \vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\|$.

The first two properties are obvious (given the properties of real numbers). The last one requires proof. At this point we introduce a very useful notion : The dot product.
$\vec{x}.\vec{y} = x_1y_1 + x_2y_2 + \ldots$. The dot product obeys the following rules :

1. $\vec{x}.\vec{x} \geq 0$ with equality if and only if $\vec{x} = 0$. (This is because $\vec{x}.\vec{x} = \|\vec{x}\|^2$.
2. $\vec{x}.\vec{y} = \vec{y}.\vec{x}$.
3. $\vec{x}.(\vec{y} + \vec{z}) = \vec{x}.\vec{y} + \vec{x}.\vec{z}$.
4. $(a\vec{x}).\vec{y} = \vec{x}.(a\vec{y}) = a\vec{x}.\vec{y}$.

One can prove the extremely important

Lemma 2.1. *Cauchy-Schwarz inequality :* $|\vec{x}.\vec{y}| \leq \|\vec{x}\|\|\vec{y}\|$ *with equality if and only if* $\vec{x} = a\vec{y}$ *for some real* a.

Proof. The only inequality about dot products that we know is the trivial $\vec{x}.\vec{x} \geq 0$. How can we use this to prove such a non-trivial result ? The technique is called "arbitrage" or sometimes "amplification". You somehow introduce a parameter t into the "trivial" inequality that you know. Since we need to prove something that involves $\vec{x}$ and $\vec{y}$ the only thing we can do is to say $(\vec{x}+t\vec{y}).(\vec{x}+t\vec{y}) \geq 0 \; \forall \, t$. Therefore $\|\vec{x}\|^2 + 2t\vec{x}.\vec{y} + t^2\|\vec{y}\|^2 \geq 0 \; \forall \, t$. Since the inequality is true for all t, let's choose that value of t that minimises the left hand side (i.e. its derivative w.r.t t is zero). Doing so gives the Cauchy-Schwarz inequality. □

Note that for $\mathbb{C}^n$, the dot product is defined as $\vec{z}.\vec{w} = z_1\bar{w}_1 + z_2\bar{w}_2 + \ldots$. The Cauchy-Schwarz and triangle inequalities can be proved for this too. (In exactly the same way.)

3 Topology

One small piece of notation (that you might have seen before) : If $f : A \to B$ is a function, and $E \subset B$ is a subset, then $f^{-1}(E)$ is the set of all elements x of A such that $f(x) \in E$. This is called the inverse image of E.

Notice anything common in the following situations ?

1. One can measure the "distance" between rationals using the absolute value function.
2. Likewise, the distance between points in $\mathbb{R}^n$ can be measured using the norm. (This is called the "Euclidean" space.)
3. Consider the grid $\mathbb{Z} \times \mathbb{Z}$ (a map of New York city if you like). If you want to travel from one point to another in a taxi, then the (shortest) distance you need to travel is certainly not the usual Pythagorean distance. It is $Distance((x, y), (a, b)) = |x - a| + |y - b|$. (In Vizag it is safe to add an extra amount based on how naive you are. But Vizag, being my native place, is awesome.)

In all of the above, there is a way to measure distance. (It is not the same thing necessarily in all the examples.) All the distances satisfy some "reasonable" properties :

1. $d(P, Q) = d(Q, P)$.
2. $d(P, Q) \geq 0$ with equality if and only if $P = Q$.
3. $d(P, Q) \leq d(P, R) + d(R, Q)$.

We define any set X equipped with a "distance" function $d : X \times X \to \mathbb{R}$ satisfying the above properties to be a *metric space*. Why care about making such a definition ? Because in a metric space, you can talk about sequences, convergence, continuous functions, etc, i.e., most of the machinery used to handle real numbers goes through. So potentially, one can hope to answer interesting questions like "If I have a function $f : X \to X$, does it have a fixed point ?"

From now onwards, at least for a few classes, we will study the general properties of metric spaces. (The amount of effort to prove these properties for the special case of real numbers is not much less. So might as well prove things in general.)

Before we proceed further, let's "recall" (actually we will do this later from "scratch" anyway, but even then, let's cheat) the following observations :

1. The function $f(x) = 1$ when $x = 1$ and $f(x) = 2x^2$ when $0 \leq x < 1$ does not achieve a maximum on $[0, 1]$.

2. The function $f(x) = \frac{1}{x}$ on $(0, 1)$ is continuous, yet does not achieve a maximum.

3. The set $\{1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots\}$ "converges" to 0 and yet does not contain it.

These examples illustrate that even in the case of just real numbers, things can go wrong if you work over the wrong sort of sets. This motivates us to define open sets, closed sets, etc in the general context of metric spaces. (By the way, you can define open sets, closed sets, etc even more generally, i.e., without a metric. That is the subject of Topology.)

How can one define an open set ? Well why is $(0, 1)$ open but $(0, 1]$ not ? Roughly, the latter has a "boundary" point. What is a boundary point ? Something that "touches" both the "inside" and "outside" of the set. Sounds circular does it not ? So let's start this way by defining the most basic example of an open set :
An open ball of radius $r > 0$ centred at P $B_r(P)$ in a metric space (X, d) is the set of all $x \in X$ such that $d(x, P) < r$. (So for example, an open "ball" in $\mathbb{R}$ equipped with the absolute value metric is simply an open interval.)
A closed ball of radius r centred at P is $x \in X$ such that $d(x, P) \leq r$.

Metric spaces can be weird as Finland (expletive deleted). For example, let X be the finite set $\{a, b, c\}$. Here is a metric on it : $d(a, a) = d(b, b) = d(c, c) = 0$ and the distance between any two distinct objects is 1. This is not that weird. (After all, the vertices of an equilateral triangle satisfy this.) But you can put exactly the same metric on $\mathbb{R}$!! That is, the distance between any two distint real numbers is 1 (and $d(x, x) = 0$)! (Since I like name-dropping, this metric induces what is known as "the discrete topology" by the way, where *every* set is open. Ignore this comment.)
So this weird example also shows us one more thing, that it is not true that the smallest closed set containing the open ball of radius r is necessarily the closed ball of radius r!! (For the weird metric, the smallest closed set is the open ball of radius 1 itself because the blasted thing is also closed! The closed ball of radius 1 is too big! It is the entire set X itself!)

Another piece of terminology : A closed rectangle $R \subset \mathbb{R}^n$ is $[a_1, b_1] \times [a_2, b_2] \times \ldots$. An open rectangle is $(a_1, b_1) \times (a_2, b_2) \times \ldots$.

Before we proceed to defining open sets, closed sets, etc, here is one last but useful definition : A set $E \subset \mathbb{R}^n$ is said to be convex if whenever $P, Q \in E$, the line segment joining them, i.e., $tP + (1 - t)Q$ where $0 \leq t \leq$ is also contained in E. For example, balls

(whether open or closed) and rectangles are convex.
Let's make the following definitions for a metric space (X, d) :

1. A neighbourhood $N_r(p)$ (sometimes called an "open" neighbourhood) is the open ball of radius r centred at p.

2. An interior point of a set E is the set of all points p such that there is a neighbourhood $N_r(p)$ that is completely contained in E. The interior (denoted as $Int(E)$ or E^o) is defined as the set of all interior points of E.

3. An open set is one where every point is an interior point, i.e., $E = Int(E)$.

4. A closed set E is one whose complement E^c is open.

5. A set E is called bounded if there is a point $q \in X$ and a number $R > 0$ such that $d(p, q) < R \; \forall \; p \in E$.

6. A limit point of a set E is a point $p \in X$ (not necessarily in E) such that *every* neighbourhood $N_r(p)$ contains a point $q_r \neq p$ such that $q_r \in E$. In fact, you can prove that every neighbourhood must contain infinitely points $\in E$. So a finite set has no limit points.
   When we define sequences and limits of sequences in a metric space (you can guess what their definitions are), you can see for yourself that a limit point of a set E is simply the limit of a (non-trivial) sequence of points from E.

7. An isolated point p of a set E if $p \in E$ but p is not a limit point.

To be cont'd.....