# Notes for 16 Mar (Thursday)

## 1 Recap

1. Proved that $\int F' = F(b) - F(a)$, integration-by-parts for Riemann integrals as well as for RS integrals. (The latter explains the symmetry between $\alpha$ and $f$.)

2. Defined integration of vector-valued functions. Proved that $\| \displaystyle\int \vec{f} d\alpha \| \leq \displaystyle\int \|\vec{f}\| d\alpha$.

3. Defined the notion of length of a curve and rectifiable curves. Proved that if $\gamma$'s derivatives are continuous then its length is $\displaystyle\int_a^b \|\gamma'\| dt$.

## 2 Sequences and series of functions

Suppose we have a sequence of functions (for example, suppose we are trying to solve a differential equation by means of an iterative procedure on a computer) $f_n(x) : E \subset \mathbb{R} \to \mathbb{C}$. (By the way, most of what I will say in this setting goes through for metric spaces.) What should it mean for $f_n \to f$ for some function $f$ ? Likewise, if one consider $\sum f_n$ what does it mean for it to converge to some function $g$ ?
The simplest notion is that of "pointwise convergence". This means that $f(x) = \lim_{n\to\infty} f_n(x)$ for all $x \in E$. But this notion is not good enough for most purposes. For instance, suppose $f_n(x)$ are all continuous, i.e., $\lim_{y\to x} f_n(y) = f_n(x)$. Then is $f(x)$ also continuous? i.e., is $\lim_{y\to x} f(y) = \lim_{y\to x} \lim_{n\to\infty} f_n(y) = \lim_{n\to\infty} \lim_{y\to x} f_n(y) = f(x)$ ? Unfortunately, the order of limits, differentiation, and integration cannot be interchanged in general. The following counterexamples demonstrate this phenomenon.

1. Let $s_{m,n} = \frac{m}{m+n}$. Then if you fix $n$, $\lim_{m\to\infty} s_{m,n} = 1$. Thus $\lim_{n\to\infty} \lim_{m\to\infty} s_{m,n} = 1$. But the other limit is 0.

2. Take $f_n(x) = x^n$ on $[0, 1]$. Let $f(x) = 0$ when $x < 1$ and $f(1) = 1$. $f_n(x) \to f(x)$ pointwise. $f_n(x)$ is continuous but $f$ isn't.

3. Let $f_m(x) = \lim_{n\to\infty} (\cos m!\pi x)^{2n}$. Note that if $m!x$ is an integer, then $f_m(x) = 1$, otherwise it is 0. Note that $f_m$ are Riemann integrable. Let $f(x) = \lim_{m\to\infty} f_m(x)$. Note that if $x$ is a rational then $f(x) = 1$. Otherwise, it is 0. This function is discontinuous everywhere. It is not Riemann integrable (U-L=1 for any partition.)

4. Let $f_n(x) = \frac{\sin(nx)}{\sqrt{n}}$. Note that $f_n(x) \to f(x)$ where $f(x) = 0$ for all $x$. $f_n'(x) = \sqrt{n}\cos(nx)$. When $x = 0$ it goes to $\infty$.

5. Let $f_n(x) = n^2 x(1-x^2)^n$. Note that $f_n(x) \to f(x)$ where $f(x) = 0$ for $|x| \leq 1$ by a theorem we proved earlier (or L'Hospital if you prefer). Note that $\int_0^1 f(x)dx = 0$. Now $\int_0^1 f_n(x)dx = \frac{n^2}{2n+2}$ which goes to $\infty$.

# 3 Uniform convergence

Clearly what is going wrong in the above examples is the following : Suppose you take $s_{n,m}$, it is true that as $m \to \infty$, $s_{n,m}$ gets close to some number $a_n$ but the problem is how fast is it getting close ? In particular, does the speed of convergence depend on $n$?

To fix these issues, we define the notion of uniform convergence. We say that a sequence of functions $f_n : E \to (X, d)$ converges uniformly to $f$ if for every $\epsilon > 0$ there exists an $N$ such that $n > N$ implies that $d(f_n(x), f(x)) < \epsilon$ for all $x \in E$. In particular, this implies that $f_n(x) \to f(x)$ for every $x$, i.e., $f_n$ definitely converge pointwise. Likewise, we say that a series $\sum f_i(x)$ of real-valued functions converges uniformly to $s(x)$ if for every $\epsilon > 0 \; \exists \; N$ s.t. $n > N \;\Rightarrow\; |\sum_{i=1}^{n} f_i(x) - s(x)| < \epsilon \; \forall \; x \in E$. From now onwards, we shall deal only with real-valued functions although many of these concepts readily generalise to metric spaces.

There is a Cauchy criterion for uniform convergence too :

**Theorem 1.** *The sequence of functions $f_n(x)$ defined on $E$ converges uniformly o $E$ iff for every $\epsilon > 0$ there is an $N$ such that $m, n > N$ implies that $|f_m(x) - f_n(x)| < \epsilon$ for all $x \in E$.*

*Proof.* Suppose $f_n$ converges uniformly to $f$. Then choose $N$ s.t $n > N$ implies that $|f_n(x) - f(x)| < \epsilon/2$. Thus $|f_m(x) - f_n(x)| \leq |f_n(x) - f(x)| + |f_m(x) - f(x)| < \epsilon$.
Suppose the other way round holds. Then at a fixed $x$, since $a_n = f_n(x)$ is a Cauchy sequence, it converges to a number $f(x)$. Now choose $N$ s.t. $|f_m(x) - f_n(x)| < \epsilon$ for all $x \in E$ and $n, m > N$. Now $|f(x) - f_n(x)| \leq |f(x) - f_m(x)| + |f_m(x) - f_n(x)| < |f(x) - f_m(x)| + \epsilon$ for all $x \in E$ and $n, m > N$. Now assume that $n < m$. Fix $n$, $x$ and tend $m \to \infty$ to see that $|f(x) - f_n(x)| < \epsilon \; \forall \; x \in E$ *and* $n > N$. This means that $f_n$ converges to $f$ uniformly. □

The following obvious criterion is useful - Suppose $f_n$ converges to $f$ pointwise. Then $M_n = \sup_{x \in E} |f_n(x) - f(x)|$ converges to 0 iff $f_n$ converges to $f$ uniformly.

There is a nice test for uniform convergence of a series of functions. It is the Weierstrass $M$-test.

**Theorem 2.** *Suppose $f_n$ is a sequence of functions defined on $E$, and suppose $|f_n(x)| \leq M_n \; \forall \; x \in E$. Then $\sum f_n$ converges uniformly if $\sum M_n$ converges.*

*Proof.* Indeed if $\sum M_n$ converges, then it is a Cauchy sequence and hence $\sum_{i=n}^{m} M_i < \epsilon$. Thus $|\sum_{i=n}^{m} f_n(x)| \leq \sum_{i=n}^{m} |f_n(x)| < \sum_{i=n}^{m} M_i < \epsilon \; \forall \; x \in E$. By the Cauchy criterion this means that $\sum f_i(x)$ converges uniformly on $E$. □

# 4 Uniform convergence and continuity

**Theorem 3.** *Suppose $f_n \to f$ uniformly on a set $E$ in a metric space. Let $x$ be a limit point of $E$ and suppose $\lim_{t\to x} f_n(t) = A_n$. Then $A_n$ converges and $\lim_{t\to x} f(t) = \lim_{n\to\infty} A_n$. That is, $\lim_{t\to x}\lim_{n\to\infty} f_n(t) = \lim_{n\to\infty}\lim_{t\to x} f_n(t)$.*

*Proof.* Suppose $N$ is s.t. $m, n > N$ implies that $|f_n(t) - f_m(t)| < \epsilon\ \forall\ t \in E$. Take $t \to x$ to get $|A_n - A_m| < \epsilon$. Thus $A_n$ is Cauchy and hence converges to some real number $A$. Now assume that $M$ is so large that $n > M$ implies that $|f_n(t) - f(t)| < \epsilon\ \forall\ t \in E$ and that $|A_n - A| < \epsilon$. In other words, $f_n(t) - \epsilon \leq f(t) \leq f_n(t) + \epsilon\ \forall\ t \in E$. Now suppose $t_k$ is any sequence converging to $x$. Then taking limits, $A_n - \epsilon \leq \liminf f(t_k) \leq \limsup f(t_k) \leq A_n + \epsilon$. Now taking $n \to \infty$ we see that $A - \epsilon \leq \liminf \leq \limsup \leq A + \epsilon$. Thus as $\epsilon \to 0$, $A = \lim f(t_k)$. Since $t_k$ is arbitrary, $\lim_{t\to x} f(t) = A$. □

This implies that if $f_n$ is a sequence of continuous functions converging uniformly, then the limit is continuous. But the converse is not true. Then example $f_n(x) = n^2x(1-x^2)^n$ does not converge uniformly. (This will see later on because if it did converge uniformly then you can interchange integration and limits.) However the limit is continuous.