Notes for 17 Mar (Friday)

1 Recap

1. Defined pointwise convergence. Gave a number of examples to illustrate that pointwise convergence is not a great concept. In particular, interchanging limits with other operations of calculus does not work for pointwise convergence.

2. Defined uniform convergence. Proved the Cauchy criterion for it.

3. Proved the Weierstrass M-test for the uniform convergence of a *series* of functions.

4. Proved that uniform limits of continuous functions are continuous. (But the converse is not true.)

2 Uniform convergence and continuity

Theorem 1. *Suppose K is compact, and*

1. *f_n is a sequence of continuous functions on K.*

2. *$f_n \to f$ pointwise where f is continuous.*

3. *$f_n(x) \geq f_{n+1}(x)$.*

Then $f_n \to f$ uniformly.

Proof. Let $g_n = f_n - f$. Let $K_n \subset K$ be all $x \in K$ s.t. $g_n(x) \geq \epsilon$. Since g_n is continuous, K_n is closed and hence compact. Since $g_n \geq g_{n+1}$, $K_{n+1} \subset K_n$. Fix $x \in K$. Since $g(x) \to 0$, $x \notin \cap K_n$. But if $\cap K_n$ is empty, then at least one of the K_N is empty. Therefore $0 \leq g_n(x) < \epsilon \ \forall \ x \in K$ and $n \geq N$. □

Compactness is essential. For example, if $f_n(x) = \frac{1}{nx+1}$ on $(0,1)$, then $f_n(x) \to 0$ monotonically but the convergence is not uniform. Why not? If $|f_n(x)| < \epsilon$ then $n > \frac{\frac{1}{\epsilon}-1}{x}$ which of course goes to infinity as $x \to 0$.

Now we make a very important definition. Suppose X is a metric space, let $\mathcal{C}(X)$ be the set of all complex-valued continuous bounded functions on X. (It is a huge set.) We shall make $\mathcal{C}(X)$ into a metric space!! Indeed, define $\|f\| = \sup_{x \in X} |f(x)|$. This is finite by assumption. It is 0 if and only if $f = 0$. Note that $\sup |(f+g)| \leq \sup |f| + |g| \leq$

$\sup|f| + \sup|g|$. Thus $d(f,g) = \|f-g\|$ is indeed a valid metric.

One of our theorems can be written as $f_n \to f$ in $\mathcal{C}(X)$ if and only if f converges uniformly on X. The nice thing is that the above metric makes $\mathcal{C}(X)$ into a complete metric space.

Proof. Suppose f_n is a Cauchy sequence in $\mathcal{C}(X)$. We need to prove that it converges to some continuous bounded function $f(x)$ in the $\mathcal{C}(X)$ metric.
Indeed, $\sup|f_n(x) - f_m(x)| < \epsilon$ for all $n, m > N$ means that by the Cauchy criterion, $f_n(x)$ uniformly converges to a function $f(x)$. Since uniform limits of continuous functions is continuous, so is $f(x)$. Now $|f(x) - f_n(x)| < \epsilon$ for $n > N$. Therefore $|f(x)| < M_n$ for all x since $f_n(x)$ is. Therefore f is bounded and continuous. Since $f_n \to f$ uniformly, it does so in $\mathcal{C}(X)$. □

3 Uniform convergence and integration

The bottom line is that uniform limits of RS functions is RS.

Theorem 2. *Let $\alpha : [a,b] \to \mathbb{R}$ be monotonically increasing. Suppose $f_n : [a,b] \to \mathbb{C}$ is RS w.r.t α and $f_n \to f$ uniformly on $[a,b]$. Then f is RS and $\displaystyle\int_a^b f d\alpha = \lim_{n\to\infty} \int_a^b f_n d\alpha$.*

Proof. Suppose $n \geq N$ is so large that $|f_n(x) - f(x)| < \epsilon \ \forall\ x \in [a,b]$. Choose a partition P such that $U(P, f_N, \alpha) - L(P, f_N, \alpha) < \epsilon$. Now $f_N(x) \in [m_i, M_i]$ when $x \in [x_{i-1}, x_i]$. Therefore $f(x) \in [m_i - \epsilon, M_i + \epsilon]$ for such x. Thus $U(P, f, \alpha) - L(P, f, \alpha) < \sum(M_i - m_i + 2\epsilon)\Delta\alpha_i < \epsilon + 2\epsilon(\alpha(b) - \alpha(a)) = \tilde{\epsilon}$ which can be made arbitrarily small. Thus f is RS. Moreover, since $|\sum f(t_i)\Delta\alpha_i - \displaystyle\int_a^b f d\alpha| < \tilde{\epsilon}$ for any choice of $t_i \in [x_{i-1}, x_i]$ and likewise $|\sum f_N(t_i)\Delta\alpha_i - \displaystyle\int_a^b f_N d\alpha| < \epsilon$, we see that $|\displaystyle\int_a^b f d\alpha - \int_a^b f_N d\alpha| < \tilde{\epsilon} + \epsilon(\alpha(b) - \alpha(a)) + \epsilon = \hat{\epsilon}$. This means that $\displaystyle\int f d\alpha - \hat{\epsilon} \leq \int f_N d\alpha \leq \int f d\alpha + \hat{\epsilon}$. As usual, this means (after using lim sup and lim inf by tending $N \to \infty$, and then $\epsilon \to 0 \Rightarrow \hat{\epsilon} \to 0$ we get $\displaystyle\int f d\alpha = \lim \int f_N d\alpha$. □

As corollary, we see that if f_n are RS and $\sum f_n(x)$ converges uniformly to $f(x)$ then f is RS and $\displaystyle\int_a^b f d\alpha = \sum_{n=1}^{\infty} \int_a^b f_n d\alpha$.

4 Uniform convergence and differentiation

Taking the example $f_n(x) = \frac{\sin(nx)}{\sqrt{n}}$ we see that f_n converges uniformly to 0. Indeed, $|f_n(x)| \leq \frac{1}{\sqrt{n}} < \epsilon$ for all x when $n > \frac{1}{\epsilon^2}$. However f_n' does not converge to f'. So clearly we need more than uniform convergence here. In fact, differentiation is a harder operation

theoretically than integration. (Although for calculations, integration is obviously way harder.)
Here is the first version of the theorem we are referring to. This version has the advantage of having a simple(r) proof. Then we will deal with Rudin's more general version.

Theorem 3. *If $f_n : (a-\epsilon, b+\epsilon) \to \mathbb{R}$ is a sequence of functions whose derivatives exist and are continuous on $[a,b]$. Suppose $f_n(x_0)$ converges to some number A for some point $x_0 \in [a,b]$. If $f_n^{'}$ converges uniformly on $[a,b]$ to a function g then f_n converge uniformly on $[a,b]$ to f such that $f^{'}(x) = \lim_{n\to\infty} f_n^{'}(x)$.*

Proof. Firstly, by the fundamental theorem of calculus,

$$f_n(x) = f_n(x_0) + \int_{x_0}^{x} f_n^{'}(t)dt \tag{1}$$

Now as $|f_n(x) - f_m(x)| \leq \int_{x_0}^{x} |f_n^{'}(t) - f_m^{'}(t)|dt$. Since $f_n^{'}$ uniformly converge, they are uniformly Cauchy. This means that $|f_n(x) - f_m(x)| < \epsilon|x - x_0| \ < \epsilon(b-a) \forall\, n, m > N$ and all $x \in [a,b]$. This means that f_n are uniformly Cauchy and hence f_n converge uniformly to some function f.
Taking the limit as $n \to \infty$ in equation 1 we see that $f(x) = A + \int_{x_0}^{x} g(t)dt$ by the previous theorem on interchange of integrals and limits. Of course this implies that $A = f(x_0)$. By the earlier theorems, g is continuous and hence by the fundamental theorem of calculus, $f^{'}(x) = g(x)$. □

Now let us drop the assumption of continuity of $f_n^{'}$.

Theorem 4. *If $f_n : (a-\epsilon, b+\epsilon) \to \mathbb{R}$ is a sequence of functions whose derivatives exist on $[a,b]$. Suppose $f_n(x_0)$ converges to some number A for some point $x_0 \in [a,b]$. If $f_n^{'}$ converges uniformly on $[a,b]$ to a function g then f_n converge uniformly on $[a,b]$ to f such that $f^{'}(x) = \lim_{n\to\infty} f_n^{'}(x)$.*

Proof. The only way to get derivatives into the picture without using the fundamental theorem of calculus is to use the MVT.
Note that there exists a $t_n \in [x_0, x]$ (or if $x < x_0$, then $[x, x_0]$) such that $f_n(x) - f_n(x_0) + f_m(x_0) - f_m(x) = (f_n^{'} - f_m^{'})(t_{n,m})(x - x_0)$. Now $|f_n(x) - f_m(x)| \leq |f_n(x) - f_n(x_0) + f_m(x_0) - f_m(x)| + |f_n(x_0) - f_m(x_0)| < \frac{\epsilon}{2} + |f_n^{'}(t_{n,m}) - f_m^{'}(t_{n,m})|(b-a)$ where $m, n > N$ such that $|f_n(x_0) - f_m(x_0)| < \frac{\epsilon}{2}$. Since $f_n^{'}$ converge uniformly, they are uniformly Cauchy and hence you can assume that N is so large that $n, m > N$ implies that $|f_n^{'}(t) - f_m^{'}(t)| < \frac{\epsilon}{2(b-a)}$ for all t. Thus $|f_n(x) - f_m(x)| < \epsilon$ for all $n, m > N$. This means that f_n is uniformly Cauchy and hence converges uniformly to a continuous function f on $[a,b]$.
The proof will be continued the next time □