Notes for 19th Jan (Thursday)

1 Recap

1. We finished our discussion of Euclidean space ($\mathbb{R}^n$) and proved the Cauchy-Schwarz inequality. In a moment of weakness I succumbed and proved the AM-GM inequality (which has nothing to do with this course).

2. We defined metric spaces and discussed the weird example of the discrete space (where every pair of distinct points are at unit distance from each other). We saw that *every* point is an open ball. (More on this today.)

3. We defined open sets, closed sets, interior, neighbourhood, bounded sets, and limit points of a set E (every neighbourhood of p contains a point from E that is distinct from p). We proved that if p is a limit point of E then *every* neighbourhood actually contains *infinitely* many points of E. (Thus finite sets cannot have limit points.) Lastly, we defined isolated points. (Those sad idiots who are points of E but are not limit points.)

2 Topology (cont'd...)

1. E is said to be dense in X if every point of X is either a point of E or a limit point of E. (That is, every point in X can be approximated using points of E. For example, rationals are dense in reals (thus the name "the density property").)

2. E is said to be perfect if E is closed and every point of E is a limit point of E. (What is an example of something closed but not perfect ?)

3. The closure of a set E is the *smallest* closed set containing E (that is every closed set containing E also contains its closure). The closure of E is denoted as $\bar{E}$. The closure of a closed set is itself.

4. The boundary of a set $Bd(E) = \bar{E} - Int(E)$.

We now prove a few intuitive but useful results. (Remember that while it is okay in this class to rely on your intuition about real numbers to deal with metric spaces (we will almost exclusively deal with the real numbers and their usual metric. Even if we deviate, it will still be "intuitive" things but not weird stuff), in the topology class, it will not be so. For example, the weird discrete metric space we discussed is very counterintuitive.)

1. Every neighbourhood is an open set.
   Pf : Indeed, given a point q in $N_r(p)$, by the triangle inequality, $q \in N_{\frac{r-d(p,q)}{2}}(q) \in E$.

2. Let E_α be an arbitrary collection of subsets of X. Then $(\cup_\alpha E_\alpha)^c = \cap_\alpha E_\alpha^c$. This is just an easy set theory statement but is useful in general. (De Morgan's law.)

3. A set E is closed if and only if it contains all its limit points.
   Pf : Assume E is closed. Let p be a limit point of E. Suppose $p \notin E$. This means that $p \in E^c$ which is open by definition. Therefore there is a neighbourhood of p that is completely contained in E^c. But this contradicts the assumption that p is a limit point of E.
   Assume that E contains all its limit points. We shall prove that E^c is open. Indeed, if $p \in E^c$ such that no matter what neighbourhood $N_r(p)$ we choose, it is not wholly contained in E^c, i.e., $\exists q_r \in E \cap N_r(p)$, then by definition p is a limit point of E and therefore cannot be in E^c.

4. For any collection of open sets G_α their union is open. For any collection of closed sets F_α their intersection is closed. For a *finite* collection of open sets $G_1, G_2, \ldots, G_n$ their intersection is open. Likewise finite unions of closed sets is closed.
   Why finite ? (Hint: Look at the sets $G_n = (-1/n, 1/n)$.
   Pf : We will prove only the statements about open sets because the corresponding ones for closed sets follow by taking complements and using De Morgan's law.
   Indeed, if you take any arbitrary collection G_α of open sets and if $p \in \cup_\alpha G_\alpha$ this means $p \in G_\beta$ for some β. Therefore, $N_p(r) \subset G_\beta \subset \cup_\alpha G_\alpha$.
   If G_i are finitely many open sets and $p \in \cap_i G_i$ then let $r = min(r_1, \ldots, r_n)$ where r_i is the radius of a neighbourhood contained in G_i. Then $N_r(p) \subset \cap_i G_i$.

5. If E is a set and $E^{'}$ be the set of all limit points of E then $G = E \cup E^{'}$ is closed and in fact is the closure of E.
   Pf : If $x \in G^c$ then we claim that there exists an $r > 0$ such that $B_r(x) \subset G^c$. If not, then every neighbourhood $B_r(x)$ of x has a point $q_r \neq x$ in G. Note that either $q_r \in E$ or $q_r \in E^{'}$. If $q_r \in E^{'}$ then since q_r is a limit point of E, the open set $B_r(x)$ contains a point $\tilde{q}_r \neq q_r$ from E. Therefore, the bottom line is that every neighbourhood of x contains a point from E (obviously $\neq x$ because $x \notin E$). This means that x is a limit point, i.e., $x \in E^{'} \subset G$. This is a contradiction. Hence G^c is open and thus G is closed.
   If any closed set F contains E, then suppose x is a limit point of E. We will prove that $x \in F$ thus proving that $G \subset F$ (hence $G = \bar{E}$ by definition). Indeed, if $x \notin F$ then because F^c is open (by definition), there is a neighbourhood $B_r(x) \subset F^c$. But by definition of a limit point, that neighbourhood has a point $q_r \neq x$ from E. This is a contradiction.

6. Let $E \subset \mathbb{R}$ be a non-empty set of reals that is bounded above. By the least upper bound property, $y = \sup_E$ exists. We claim that indeed $y \in \bar{E}$.
   Pf : Since y is the sup, for any $\epsilon > 0$ $y - \epsilon$ is not an upper bound, i.e., there is an $x_\epsilon \in E$ such that $y \geq x_\epsilon > y - \epsilon$. This means that every neighbourhood $(y-\epsilon, y+\epsilon)$ of y contains a point $x_\epsilon \in E$. By definition y is a limit point of E and hence $y \in \bar{E}$.

Before we proceed further, note the following ambiguity : Is $(-1, 1)$ an open set ? Well, this question by itself does not make sense. Are we asking whether $(-1, 1)$ is an open set of $\mathbb{R}$ of $\mathbb{R}^2$ or something else ? (That is what metric space are we considering?) It is an open subset of $\mathbb{R}$ (Why? This is because given any point $x \in (-1, 1)$ the neighbourhood $(x-(1-|x|), x+(1-|x|)) \subset (-1, 1)$.) But it is not an open subset of $\mathbb{R}^2$ (Why? because no disc is contained in a line). These considerations motivate the following definition.

Suppose $E \subset Y \subset X$ where (X, d) is a metric space. Note that $(Y, d|_{Y \times Y})$ is a metric space in its own right. So the questions "Is E open in X?" and "Is E open in Y?" may not have the same answer. (As $(-1, 1)$ illustrates.) We say that E is open *relative to* Y if whenever $p \in E$, there exists an $r > 0$ such that the set of all points $y \in Y$ such that $d(y, p) < r$ is completely contained within E - This will be called "a neighbourhood of p in Y". Here is a nice theorem that tells us more about this notion - To be cont'd...