Notes for 1 Mar (Wednesday)

1 Recap

1. Finished the proof of Riemann's rearrangement theorem.
2. Defined limits and continuity.
3. Wrote an alternate characterisation of continuity (inverse images of open sets are open). Proved that continuous functions that compact sets to compact sets and that $1-1$ continuous functions on compact sets have continuous inverses.

2 Uniform continuity

If you take a more advanced course in analysis or perhaps topology, something called the Arzela-Ascoli theorem will be proved. We will set up the ground work with one last definition.

Note that in the definition of continuity we said that for every $\epsilon > 0$, there exists a $\delta > 0$ such that $d(x,p) < \delta \Rightarrow d(f(x), f(p)) < \epsilon$. However, if we change p, the δ might change, i.e., δ might depend on p.

Not convinced ? Here is an example : Suppose we take $f(x) = \frac{1}{x}$ on $(0,1)$. Fix $\epsilon > 0$. Suppose a single $\delta > 0$ made sure that $|x-p| < \delta \Rightarrow |\frac{1}{x} - \frac{1}{p}| < \epsilon$. Then $|x-p| < |x||p|\epsilon < (\delta + |p|)|p|\epsilon$. Now choose $|p|$ to be so small that the right hand side is less than $\frac{\delta}{3}$. Now surely, $p + \frac{\delta}{3}$ will lead to a contradiction.

So we need a new definition for these kinds of situations. It is called Uniform continuity. Indeed, a function $f : X \to Y$ is said to be uniformly continuous if for every $\epsilon > 0$ there exists a $\delta > 0$ (possibly depending on ϵ but not on anything else) such that whenever *any* two points p and q in X satisfy $d(p,q) < \delta$, then $d(f(p), f(q)) < \epsilon$. The above example shows that $\frac{1}{x}$ is NOT uniformly continuous (despite being continuous) on $(0,1)$. In fact, this can be generalised. On any bounded noncompact set $E \subset \mathbb{R}$ you can construct a function which is continuous but not uniformly so. So when are continuous functions uniformly continuous ?

Theorem 1. *Suppose X is compact. Then any continuous function $f : X \to Y$ is uniformly continuous.*

Proof. Indeed, suppose we are given $\epsilon > 0$. Let $\delta_q > 0$ be such that $d(f(p), f(q)) < \frac{\epsilon}{2}$ whenever $d(p,q) < \delta_q$. Now consider the balls $B_q = B_{\delta_q/2}(q)$ where q ranges over all of X. Since X is compact, you need finitely many balls $B_{q_1}, B_{q_2}, \ldots, B_{q_n}$ to cover X. Now let δ be the smallest of $\frac{\delta_{q_i}}{2}$. So given any two points p, q, surely q is in one of the B_{q_i}. Thus $d(f(q), f(q_i)) < \frac{\epsilon}{2}$. Also, since $d(p,q) < \delta \leq \delta_{q_i}/2$, this means that $d(p, q_i) < \delta_{q_i}$. Thus $d(f(p), f(q_i)) < \frac{\epsilon}{2}$. Likewise, $d(f(q), f(q_i)) < \frac{\epsilon}{2}$. Thus $d(f(p), f(q)) < \epsilon$. □

3 Continuity and connectedness

Akin to compactness, continuous functions preserve connectedness, i.e.,

Theorem 2. *Suppose $f : E \subset X \to Y$ is continuous, and E is connected, then $f(E)$ is connected.*

Proof. Suppose $f(E) = A \cup B$ where $A = U_1 \cap E \neq \phi$ and $B = U_2 \cap E \neq \phi$ are relatively open in E (that is, U_1 and U_2 are open in X) and are disjoint, i.e., assume that $f(E)$ is not connected.

Since f is continuous, $f^{-1}(U_i)$ are open in X. Therefore $E = f^{-1}(f(E)) \cap E = (f^{-1}(U_1) \cap E) \cup (f^{-1}(U_2) \cap E)$ where $G = f^{-1}U_1 \cap E$ and $H = f^{-1}U_2 \cap E$ are nonempty and relatively open in E. Since A and B are disjoint, so are $f^{-1}(A) \cap E = f^{-1}(U_1) \cap E \neq \phi$ and $f^{-1}(B) \cap E = f^{-1}(U_2) \cap E \neq \phi$. This means that E is not connected, a contradiction. □

This may be used to give a simple proof of the famous intermediate value theorem.

Theorem 3. *If $f : [a,b] \to \mathbb{R}$ is continuous such that $f(a) < f(b)$, then if $c \in [f(a), f(b)]$ there exists an $x \in (a,b)$ such that $f(x) = c$.*

Proof. Indeed, we know that (a,b) is connected. Therefore $f(a,b)$ is so too. We also know that connected subsets of $\mathbb{R}$ satisfy the property that if $u, v \in E$ then $(u,v) \subset E$. This shows that $(f(a), f(b)) \subset f(a,b)$. □

You might be tempted to think that the converse is true, i.e., suppose for every $x_1 < x_2$ such that $f(x_1) < f(x_2)$ and $c \in (f(x_1), f(x_2))$ there exists an $x \in (x_1, x_2)$ such that $f(x) = c$, then f is continuous. But this is false. For example, $f(x) = \sin(1/x)$ when $x \neq 0$ and $f(0) = 0$ is not continuous but does satisfy this property.

4 Assorted topics - Discontinuities, monotonic functions, and limits at infinity

We say that $\lim_{x\to\infty} f(x) = A$ iff for every $\epsilon > 0$ there exists an $M > 0$ such that for all x satisfying $x > M$, we have $|f(x) - A| < \epsilon$. Likewise for $-\infty$.

We say that $\lim_{x\to a} f(x) = \infty$ iff for every $M > 0$ there exists a $\delta > 0$ such that $|x - a| < \delta$ implies that $f(x) > M$. Likewise for $-\infty$. Likewise for $\lim_{x\to\infty} f(x) = \infty$ and so on.

The rules of limits apply in these cases too with the understanding that $0 \times \infty, \infty - \infty, \frac{\infty}{\infty}, \frac{A}{0}$ do not make sense.

Let's define left and right hand limits to study discontinuities. Indeed, $\lim_{x\to a^+} f(x) = L$ iff for every sequence $x_n \to a$ such that $x_n > a \; \forall \, n$, $f(x_n) \to L$. Sometimes L is written as $f(a+)$. Likewise, for the left-hand limit $f(a-)$.

A function is continuous iff $f(a+) = f(a-) = f(a)$. Indeed, if a function is continuous of course this is true. Conversely, if this is true, then suppose f is discontinuous. Then, there exists an $\epsilon > 0$ such that for every integer $k > 0$ there exists an $x_k \in (a - \frac{1}{k}, a + \frac{1}{k})$ such that $|f(x_k) - f(a)| \geq \epsilon$. Suppose x_k contains infinitely many terms $x_{n_k} \geq a$ (without loss of generality). Then the subsequence $x_{n_k} \to a$ from the right and hence $f(x_{n_k}) \to f(a+)$ by assumption. This is a contradiction.

Thus for a function $f : E \subset \mathbb{R} \to \mathbb{R}$ defined at a, there are two kinds of discontinuities at a-

1. Jump discontinuities or discontinuities of the first kind - $f(a+)$ and $f(a-)$ exist. In this case, either $f(a+) \neq f(a-)$ or $f(a+) = f(a-) \neq f(a)$.
2. Discontinuities of the second kind - One of them (or perhaps both) do not exist.

Examples :

1. $f(x) = 1$ when x is rational and 0 when x is irrational is discontinuous at every point (rationals and irrationals are dense). The discontinuity is of the second kind.
2. $f(x) = x$ when x is rational and 0 when x is irrational is continuous only at 0. Everywhere else it has a discontinuity of the second kind.
3. $f(x) = \sin(1/x)$ when $x \neq 0$ and $f(0) = 0$ is discontinuous only at 0 (second kind).

If $f : (a, b) \to \mathbb{R}$ is a function such that $f(x) \leq f(y)$ if $a < x < y < b$ then f is said to be monotonically increasing. If $f(x) < f(y)$ for all such $x < y$ then it is strictly monotonically increasing. Likewise, we have monotonically decreasing functions. If either is true, the function is monotonic. Here is a result that implies there are no discontinuities of the second kind for such functions.

Theorem 4. *If $f : (a, b) \to \mathbb{R}$ is monotonically increasing, then $f(x+)$ and $f(x-)$ exist for every point $x \in (a, b)$. Moreover, $f(x+) \leq f(y-)$ for $x < y$. Furthermore, $f(x+) = \inf_{b>t>x} f(t)$ and $f(x-) = \sup_{a<t<x} f(t)$.*

Proof. On (x, b), $f(t)$ is bounded below by $f(x)$. Thus $M = \inf_{b>t>x} f(t)$ exists. Likewise, $m = \sup_{a<t<x} f(t)$ exists. For every $\epsilon > 0$, $M + \epsilon > f(t) \geq M$ for some $b > t > x$. Since f is increasing, $M + \epsilon > f(y) \geq M \; \forall \, t \geq y > x$ and hence $M + \epsilon \geq f(x+) \geq M$. Since this is true for all ϵ, $f(x+) = M$. Likewise $f(x-) = m$. Moreover, suppose $x < y$. Then for all $y > y_1 > t > x$, $f(t) \leq f(y_1) \leq f(y)$. Since $f(x+)$ is also equal to (by monotonicity of f) $\inf_{y_1>t>x} f(t)$ we see that $f(x+) \leq f(y_1)$. Therefore, $f(x+) \leq f(y-)$. $\square$