Notes for 22 Feb (Wednesday)

1 A long recap

1. We defined series $\sum a_n$ as a sum of a sequence a_n, convergence and divergence (as the convergence and divergence of the partial sums s_N). We saw that a series diverges if the corresponding sequence a_n does not go to 0 (the divergence test). We also saw that if you take a series of non-negative terms, it converges if and only if the partial sums are bounded. (Monotone bounded sequences converge.)

2. We showed the comparison test. (If you are positive and less than someone who converges, then you converge.)

3. We looked at Cauchy's theorem that in the case of non-negative decreasing terms reduced the convergence of a series to that of another series $\sum 2^n a_{2^n}$. Using this we found several examples of convergent/divergent series (like the Harmonic series and in general the p-series $\sum 1/n^p$).

4. We proved the ratio and root tests for absolute convergence, i.e. if you take $\sum a_n$ then it converges absolutely if $\limsup |a_n|^{1/n} < 1$ and diverges if the limit is larger than 1. If it is equal to 1 then you are screwed. Likewise you can also consider $\limsup \frac{|a_{n+1}|}{|a_n|}$. The root test is in theory more powerful than the ratio test. However, the ratio test is easier to use.

5. Defined power series $\sum c_n z^n$. Using the ratio/root tests we determined that a power series converges when $|z| < R$ and diverges when $|z| > R$ for a number R called the radius of convergence. $R = \frac{1}{\limsup \frac{|c_{n+1}|}{|c_n|}}$. On the circle of convergence $|z| = R$, strange things can happen. By the way, here is a small point :
$\liminf \frac{1}{|a_n|} = \frac{1}{\limsup |a_n|}$ if none of the a_n are 0, $\limsup |a_n| \neq 0, \infty$. Indeed,

$$\liminf \frac{1}{|a_n|} = \lim_{N\to\infty} \inf_{n\geq N} \frac{1}{|a_n|} = \lim_{N\to\infty} \frac{1}{\sup_{n\geq N} |a_n|}, \tag{1}$$

where the last equality is proved (recall) as follows : Assume without loss of generality that $a_n > 0$ and $b_N > 0$ where $b_N = \sup_{n\geq N} a_n$. Assume that $b_N \neq a_l$ for any l because if so, then $\frac{1}{b_N} = L$ and we are done. Let $c_N = \inf_{n\geq N} \frac{1}{a_n}$. Given a positive integer $k > 0$ there exists (inductively) an a_{n_k} such that $b_N - \frac{1}{k} \leq a_{n_k} \leq b_N$ and $n_k > n_{k-1} > \ldots$. (The last inequality holds because otherwise, either eventually all the a_n are larger than b_N which is absurd or they are eventually less than $b_N - \frac{1}{k}$. If the latter is true, then their supremum has to be less than $b_N - c$ for some $c > 0$

because the none of them is equal to the supremum and they have to get arbitrarily close to it.) Therefore $a_{n_k} \to b_N$. Likewise, we can get another subsequence $a'_{n_k} \to c_N$. Now $c_N \leq \lim_{k\to\infty} \frac{1}{a_{n_k}} = \frac{1}{b_N}$ and likewise $b_N \geq \frac{1}{c_N}$. Thus we are done.

6. Defined absolute and conditional convergence. Using summation-by-parts we proved the alternating series test. Actually we proved something more general, i.e., if b_n is a decreasing sequence that goes to 0 and the partial sums $A_N = \sum_{n=0}^{N} a_n$ form a bounded sequence then $\sum a_n b_n$ converges.

7. Defined multiplication and addition of series. Saw that the product of two convergent series need NOT be convergent. However, if one of them is absolutely convergent then it is indeed convergent. (Mertens' theorem.)

8. Defined rearrangements. Recall that if $\sum a_n$ is a series, then a rearrrangement is a new series $\sum a'_n$ such that $a'_n = a_{f(n)}$ where $f : \mathbb{Z}_+ \to \mathbb{Z}_+$ is a bijection.
Stated the Riemann rearrangement theorem and the fact that for absolutely convergent series, the rearranged series has the same sum as the original one.

2 Multiplication of series (cont'd..)

We shall now prove Mertens' theorem.

Proof. So we want to bound $\sum_{n=0}^{N} c_n - AB$ for large N. Let B_n be the partial sum $B_n = \sum_{k=0}^{n} b_k$ and likewise A_n. Indeed,

$$
\begin{aligned}
|\sum_{n=0}^{N} c_n - AB| &= |a_0b_0 + (a_1b_0 + a_0b_1) + (a_2b_0 + a_1b_1 + b_0a_2) + \ldots - AB| \\
&= |a_0(b_0 + b_1 + \ldots + b_N) + a_1(b_0 + b_1 + \ldots b_{N-1}) + \ldots - AB| \\
&= |a_0B_N + a_1B_{N-1} + \ldots + a_NB_0 - AB| \\
&= |a_0(B_N - B) + a_1(B_{N-1} - B) + \ldots - B(A - a_0 - a_1 \ldots - a_N)|, \\
&\leq |a_0(B_N - B) + a_1(B_{N-1} - B) + \ldots| + B|(A - a_0 - a_1 \ldots - a_N)| \\
&\leq |a_0(B_N - B) + a_1(B_{N-1} - B) + \ldots| + \epsilon
\end{aligned}
$$

if N is large enough. We just need to prove that $|a_0(B_N - B) + a_1(B_{N-1} - B) + \ldots| \to 0$ as $N \to \infty$. Let $\alpha = \sum_{n=0}^{\infty} |a_n|$. Note that in the sum in the last inequality, if N is very large, then first few terms of the type $B_{N-i} - B$ will be small (as long as $N - i$ is large), and in the other terms a_i will be small. More precisely speaking, choose M such that $|B_k - B| \leq \epsilon \; \forall \; k \geq M$. Then

$$
\begin{aligned}
|\sum_{n=0}^{N} c_n - AB| &\leq |a_0(B_N - B)| + |a_1(B_{N-1} - B)| + \ldots + \epsilon \\
&\leq (|a_0| + |a_1| + \ldots + |a_{N-M}|)\epsilon + |B_{M-1}a_{N-M+1} + \ldots + B_0a_N| + \epsilon
\end{aligned}
$$

Now fix M and let $N \to \infty$. Then we will see that the right-hand side goes to $\alpha\epsilon + 0 + \epsilon \ \forall \ \epsilon$. Now let $\epsilon \to 0$ to see the result. $\square$

3 Rearrangements (cont'd)..

Firstly (and thankfully), if the series is absolutely convergent, then all rearrangements converge to the same sum.

Theorem 1. *If $\sum a_n$ is a series of complex numbers that converges absolutely, then every rearrangement $\sum a_n^{'}$ converges and converges to $\sum a_n$.*

Proof. Suppose N is chosen to be so large that $\displaystyle\sum_{k=n+1}^{\infty} |a_k| < \epsilon \ \forall \ n \geq N$. Then, consider a rearrangement $a_k^{'}$. Fix N. Now choose $\tilde{N} > N$ to be so large that the sequence $a_1^{'}, \ldots, a_{\tilde{N}}^{'}$ contains $a_1, a_2, \ldots, a_N$. Now

$$|\sum_{k=1}^{\tilde{N}} a_k^{'} - \sum_{k=1}^{N} a_N| < \epsilon. \tag{2}$$

Therefore,

$$\sum_{k=0}^{N} a_N - \epsilon \leq \lim \inf_{\tilde{N}\to\infty} \sum_{k=0}^{\tilde{N}} a_k^{'} \leq \sum_{k=0}^{\tilde{N}} a_k^{'} \leq \lim \sup_{\tilde{N}\to\infty} \sum_{k=0}^{\tilde{N}} a_k^{'} \leq \sum_{k=0}^{N} a_N + \epsilon \tag{3}$$

for all N and ϵ. Now let $N \to \infty$ and then $\epsilon \to 0$ to get the result. $\square$

The above example of conditionally convergent series suggests that if a series is not absolutely convergent, then perhaps there is a rearrangement whose lim sup is perhaps different. Shockingly enough, something even stronger holds.

Theorem 2. *Let $\sum a_n$ be a convergent series of real numbers that does NOT converge absolutely. Suppose $-\infty \leq \alpha \leq \beta \leq \infty$. Then there exists a rearrangement $a_n^{'}$ with partial sums $s_n^{'}$ such that* $\limsup s_n^{'} = \beta$ *and* $\liminf s_n^{'} = \alpha$.

Proof. The proof is clever and I do not claim to have any way to simplify it.
Let $p_n = \frac{|a_n|+a_n}{2}$ and $q_n = \frac{|a_n|-a_n}{2}$, i.e., p_n consists of the positive terms of the sequence (along with 0 wherever negative terms appear) and likewise q_n. The series $\sum p_n$ and $\sum q_n$ must both diverge because otherwise, $\sum a_n$ is absolutely convergent.
Let $P_1, P_2, \ldots$ be the nonnegative terms of a_n in the order in which they occur and likewise $Q_1, Q_2 \ldots$ be the absolute values of the negative terms in their original order. Note that $\sum P_n$ and $\sum Q_n$ must diverge because they differ from $\sum p_n$ and $\sum q_n$ by zeroes.
We shall construct sequences $\{m_n\}, \{k_n\}$ such that the series

$$P_1 + \ldots + P_{m_1} - Q_1 - \ldots - Q_{k_1} + P_{m_1+1} + \ldots + P_{m_2} - Q_{k_1+1} - \ldots$$

which is clearly a rearrangement of the original series, satisfies the desired properties. Choose real-valued sequences $\alpha_n \to \alpha$ and $\beta_n \to \beta$ such that $\alpha_n < \beta_n$ and $\beta_1 > 0$.

Because $\sum P_n$ and $\sum Q_n$ diverge, the following is possible –

Let m_1, k_1 be the *smallest* integers such that

$$P_1 + P_2 \ldots + P_{m_1} > \beta_1,$$

$$P_1 + P_2 \ldots + P_{m_1} - Q_1 - Q_2 - \ldots - Q_{k_1} < \alpha_1.$$

Then let m_2, k_2 be the smallest integers such that

$$P_1 + P_2 \ldots + P_{m_1} - Q_1 - Q_2 - \ldots - Q_{k_1} + P_{m_1+1} + \ldots + P_{m_2} > \beta_2,$$

$$P_1 + P_2 \ldots + P_{m_1} - Q_1 - Q_2 - \ldots - Q_{k_1} + P_{m_1+1} + \ldots + P_{m_2} - Q_{k_1+1} - \ldots - Q_{k_2} < \alpha_2.$$

Continue this way. Suppose x_n, y_n are the partials whose last terms are P_{m_n} and Q_{k_n}. Then since $|x_n - \beta_n| \leq P_{m_n}$ and $|y_n - \alpha_n| \leq Q_{k_n}$, we see that $x_n \to \beta$ and $y_n \to \alpha$. It is clear that no other subsequential limit can be larger than β or less than α. $\square$