Notes for 22 Mar (Wednesday)

1 Recap

1. We proved that (while it is NOT true that if a sequence of continuous functions f_n converges to a continuous function f pointwise it means that f_n converges uniformly to f) if a decreasing sequence of continuous functions f_n converges to f pointwise on a compact set, then indeed f_n converges to f uniformly. Compactness of the domain was essential.

2. We defined the weird metric space $\mathcal{C}(X)$ consisting of all complex-valued bounded continuous functions on X. The metric was $d(f,g) = \sup |f(x) - g(x)|$. We proved the pleasant statement that $\mathcal{C}(X)$ is complete.

3. We proved that if a sequence of RS functions f_n converges uniformly to an RS function f, then indeed f is RS and you can interchange limits and integrals.

4. For differentiable functions we saw that uniform convergence $f_n \to f$ is NOT good enough for interchanging limits and derivatives. Instead, we proved that if $f_n^{'} \to g$ uniformly and $f_n(x_0) \to A$ for some x_0, and if $f_n^{'}$ are continuous, then indeed $f^{'} = g$. Actually we can drop the continuity assumption on $f_n^{'}$. In this more general theorem, we proved so far that $f_n \to f$ uniformly. Today we will prove that $f^{'} = g$.

2 Uniform convergence and differentiation

Theorem 1. *If $f_n : (a-\epsilon, b+\epsilon) \to \mathbb{R}$ is a sequence of functions whose derivatives exist on $[a,b]$. Suppose $f_n(x_0)$ converges to some number A for some point $x_0 \in [a,b]$. If $f_n^{'}$ converges uniformly on $[a,b]$ to a function g then f_n converge uniformly on $[a,b]$ to f such that $f^{'}(x) = \lim_{n\to\infty} f_n^{'}(x)$.*

Proof. The only way to get derivatives into the picture without using the fundamental theorem of calculus is to use the MVT.
Cont'd..... Now let $\phi_n(h) = \frac{f_n(x+h)-f_n(x)}{h}$ and $\phi(h) = \frac{f(x+h)-f(x)}{h}$. When $h \neq 0$, $\phi_n(h)$ converges uniformly to $\phi(h)$ because $|\phi_n(h) - \phi_m(h)| = \frac{|f_n(x+h)-f_m(x+h)-(f_n(x)-f_m(x))|}{|h|}$ which by earlier arguments is less than or equal to $\frac{\epsilon}{2(b-a)}$ when $n,m > N$. Thus ϕ_n converges uniformly to ϕ. Moreover, $\lim_{h\to 0} \phi_n(h) = f_n^{'}(x)$. Thus by the theorem on interchange of limits, $f^{'}(x) = g(x)$. □

Here is a surprising application of these results.

Theorem 2. *There exists a real continuous function on the real line which is nowhere differentiable. (Weierstrass.)*

Proof will be done the next time.