Notes for 22 Feb (Thursday)

1 Recap

1. We revised series.
2. Proved Merten's theorem on products of series.
3. Defined rearrangements and proved that for an absolutely convergent series, all rearrangements converge to the same sum.
4. For a conditionally convergent series, given any two numbers one can rearrange the series so that the lim sup and lim inf of the partial sums converge to the given numbers. This is Riemann's rearrangement theorem. We started proving it.

2 Riemann's theorem (cont'd)..

Let m_1, k_1 be the *smallest* integers such that

$$P_1 + P_2 \ldots + P_{m_1} > \beta_1,$$

$$P_1 + P_2 \ldots + P_{m_1} - Q_1 - Q_2 - \ldots - Q_{k_1} < \alpha_1.$$

Then let m_2, k_2 be the smallest integers such that

$$P_1 + P_2 \ldots + P_{m_1} - Q_1 - Q_2 - \ldots - Q_{k_1} + P_{m_1+1} + \ldots + P_{m_2} > \beta_2,$$

$$P_1 + P_2 \ldots + P_{m_1} - Q_1 - Q_2 - \ldots - Q_{k_1} + P_{m_1+1} + \ldots + P_{m_2} - Q_{k_1+1} - \ldots - Q_{k_2} < \alpha_2.$$

Continue this way. Suppose x_n, y_n are the partials whose last terms are P_{m_n} and Q_{k_n}. Then since $|x_n - \beta_n| \leq P_{m_n}$ and $|y_n - \alpha_n| \leq Q_{k_n}$, we see that $x_n \to \beta$ and $y_n \to \alpha$. It is clear that no other subsequential limit can be larger than β or less than α. $\square$

3 Limits and Continuity

Suppose (X, d_X) and (Y, d_Y) are metric spaces and $f : E \subset X \to Y$ is a function. Then we say that $\lim_{x \to p} f(x) = L$ if for every $\epsilon > 0$ there exists a $\delta > 0$ such that for every $x \in E$ satisfying $0 < d_X(x, p) < \delta$ we have $d_Y(f(x), L) < \epsilon$. (From now onwards we will stop using the subscripts on d.) Note that $f(p)$ need not be defined for this to make sense.

This definition can be recast in the language of sequences as follows :

$\lim_{x\to p} f(x) = L$ if and only if for every sequence $p_n \neq p$ such that $p_n \in E$ we have $\lim_{n\to\infty} f(p_n) = L$.
Proof : Indeed, if $\lim_{x\to p} f(x) = L$ it is clear that the other statement holds. If on the other hand the sequential statement holds, then suppose there exists an $\epsilon > 0$ such that no matter what δ we choose, there exists a point $x_\delta \in E \cap B_\delta(p)$ such that $d(f(x_\delta), L) \geq \epsilon$. Then choose $\delta = \frac{1}{k}$. Thus we have a sequence $x_k \in E \cap B_{1/k}(p)$ obviously converging to p such that $d(f(x_k), L) \geq \epsilon$. This is a contradiction. □

Using results from sequences we see that limits are unique and the following hold if f and g are complex-valued functions satisfying $\lim_{x\to p} f(x) = L, \lim_{x\to p} g(x) = M$.

1. $\lim_{x\to p}(f+g)(x) = L + M$.
2. $\lim_{x\to p} fg(x) = LM$.
3. If $M \neq 0$ then $\lim_{x\to p}(f/g)(x) = L/M$.
4. If $\vec{f}$ and $\vec{g}$ are functions to $\mathbb{R}^n$ then $\lim_{x\to p} \vec{f}.\vec{g}(x) = \vec{L}.\vec{M}$.

We say that $f(x)$ is continuous at a point $p \in E$ if $\lim_{x\to p} f(x) = f(p)$, i.e., for every $\epsilon > 0$ there exists a $\delta > 0$ such that for all $x \in E$ satisfying $0 < d(x,p) < \delta$ we have $d(f(x), f(p)) < \epsilon$. From this definition it is obvious that if p is an isolated point of E then every function is continuous at p.

We now consider compositions of functions. If $f : E \subset X \to Y$ and $g : f(E) \subset H \subset Y \to Z$ are two functions such that f is continuous at p, and g at $f(p)$, then the function $h : E \subset X \to Z$ defined as $h(x) = f(g(x))$ is continuous at p.
Proof : Given $\epsilon > 0$, suppose δ_1, δ_2 are chosen so that $y \in B_{\delta_2}(f(p)) \cap H$ implies that $d(g(y), g(f(p))) < \epsilon$ and $x \in B_{\delta_1}(p) \cap E$ implies that $d(f(x), f(p)) < \delta_2$, then note that $x \in B_{\delta_1}(p) \cap E$ implies that $d(g(f(x)), g(f(p))) < \epsilon$. □

The following is not just a useful characterisation of continuity but later on if you ever take topology you will see that continuity is *defined* using the following theorem.

Theorem 1. *The function $f : X \to Y$ is continuous on all of X (i.e. at every point of X) if and only if for every open set $V \subset Y$, the inverse image $f^{-1}(V) \subset X$ is open.*

Proof. Suppose f is continuous on all of X. Then, suppose $q \in V$ and $f(p) = q$ for some $p \in X$. Then since V is open, $B_\epsilon(q) \subset V$ for some small $\epsilon > 0$. By definition of continuity, there exists a $\delta > 0$ such that $d(x,p) < \delta \Rightarrow f(x) \in B_\epsilon(f(p) = q)$. Therefore, $f^{-1}(B_\epsilon(q))$ is open. This means that $f^{-1}(V) = f^{-1} \cup_{q\in V} (B_{\epsilon_q}(q)) = \cup_{q\in V} f^{-1}(B_{\epsilon_q}(q))$ is open.
Suppose the converse holds. Then given $\epsilon > 0$, $U = f^{-1}(B_\epsilon(q))$ is open. This means that since $p \in U$, there is a $\delta > 0$ such that $B_\delta(p) \subset U$. This means that f is continuous at p. □

As corollary, a map is continuous if and only if the inverse image of a closed set is closed.

Using the characterisation of limits (for isolated points these are obvious) one can prove that if f and g are continuous, then so are $f + g, fg, andf/g$ if $g \neq 0$ for the last one. Likewise, one can also prove that if you take vector valued functions $\vec{f}, \vec{g}$, they are continuous if and only if their components are continuous, and if they are continuous so are $\vec{f} + \vec{g}$ and $\vec{f}.\vec{g}$.

As examples, repeated applications of the above theorems show that polynomials and rational functions are continuous on their domains. In addition, one can easily show that $\vec{x} \to |\vec{x}|$ is continuous at every point $\vec{p}$. Indeed, given $\epsilon > 0$, choosing $\delta = \epsilon$, we see that $||\vec{x}| - |\vec{p}|| \leq |\vec{x} - \vec{p}| < \epsilon$.

4 Continuity and compactness

Here is a small definition : A map $\vec{f} : E \to \mathbb{R}^k$ is said to bounded if $|\vec{f}(x)| \leq M \; \forall \; x \in E$.

Here is an extremely important result :

Theorem 2. *If $f : E \subset X \to Y$ is continuous on E and E is compact, then $f(E)$ is a compact subset of Y.*

Proof. Suppose U_α form an open cover of $f(E)$. Then because f is continuous, $f^{-1}(U_\alpha)$ are open sets and certainly they cover E. Thus there is a finite subcover of E by $f^{-1}(U_1), f^{-1}(U_2), \ldots, f^{-1}(U_k)$. Thus $f(E) \subset U_1 \cup U_2 \ldots U_k$. Therefore $f(E)$ is compact. □

This immediately proves the so-called extreme value theorem : If $f : [a, b] \to \mathbb{R}$ is continuous, then it attains a maximum and a minimum value somewhere.
Proof : Indeed, $[a, b]$ is compact. So $f[a, b]$ is compact and is hence closed and bounded. By an earlier result, $\sup f[a, b]$ and $\inf f[a, b]$ exist (because the set is bounded) are contained in $f[a, b]$ because the set is closed. Therefore there exists $x, y \in [a, b]$ such that $f(x) = \sup f[a, b]$ and $f(y) = \in f[a, b]$.

In fact, there is nothing special about $[a, b]$. The same result holds if $[a, b]$ is replaced by a compact subset of any metric space.

Suppose we have a $1-1$ continuous function $f : X \to Y$. Then is the inverse function f^{-1} also continuous ?
Of course in general the answer is no. For example, take $f(x) = 1 + x$ when $-1 < x < 0$ and $f(x) = -1 + x$ when $0 < x \leq 1$. Then f is continuous on $(-1, 0) \cup (0, 1]$ by the laws of continuity. It is also $1 - 1$. Its inverse on the other hand is not continuous at 0. Wherein lies the problem ? That the domain is not compact. Indeed,

Theorem 3. *If $f : X \to Y$ is a continuous $1 - 1$ map and X is compact, then f^{-1} is also continuous.*

Proof. We will prove that $(f^{-1})^{-1}(Closed) = Closed$, i.e., $f(Closed) = Closed$. Indeed, closed subsets of a compact set X are compact. We know that $f(Compact) = Compact$. We also know that compact subsets of metric spaces are closed. Hence $f(Closed) = Closed$. □

Before proceeding further, here are some useful counterexamples that show how essential compactness is.

1. Note that $f(x) = x$ is continuous on all of $\mathbb{R}$ and yet is not bounded.
2. $f(x) = x$ is continuous on $(0, 1)$ and is bounded. But it does not attain a maximum.

In fact, given any noncompact set $E \subset \mathbb{R}$, there is

1. A continuous function which is not bounded : Indeed, if E is unbounded, then $f(x) = x$ does the job. If E is bounded but not closed, then let a be a limit point of E that is not in E. Then $f(x) = \frac{1}{x-a}$ is continuous (because the denominator is never zero on E). However, it is not bounded because x can get arbitrarily close to a.
2. A continuous and bounded function on E which has no maximum : If E is unbounded, then $f(x) = \frac{x^2}{1+x^2}$ is obviously bounded by 1 but its maximum is never attained. If E is bounded but not closed, then suppose a is a limit point that is not in E. Then $f(x) = \frac{1}{1+(x-a)^2}$ is continuous and bounded but never attains a maximum.