Notes for 22 Mar (Wednesday)

1 Uniform convergence and differentiation

Theorem 1. *There exists a real continuous function on the real line which is nowhere differentiable. (Weierstrass.)*

Proof. Define $\phi(x) = |x|$ on $[-1, 1]$. This has a corner at $x = 0$. The idea is to use this by scaling and translating to most other points to get a very "rough" function with lots of corners.
You can extend ϕ to all reals by making it periodic, i.e., $\phi(x+2) = \phi(x)$. Then for all s, t, $|\phi(s) - \phi(t)| \leq |s - t|$. So ϕ is continuous. Define $f(x) = \displaystyle\sum_{n=0}^{\infty} \frac{3^n}{4^n}\phi(4^n x)$. Since $0 \leq \phi \leq 1$ the Weierstrass M-test shows that this series converges uniformly. Hence f is continuous. It seems that f has too many corners to be differentiable. Indeed that is the case.
Fix a number x and a positive integer m. Let $\delta_m = \pm\frac{1}{2}4^{-m}$ where the sign is chosen so that no integer lies between $4^m x$ and $4^m(x+\delta_m)$. This can be done because the difference is $\frac{1}{2}$. Define $\gamma_n = \frac{\phi(4^n(x+\delta_m)) - \phi(4^n x)}{\delta_m}$. If $n > m$, then $4^n\delta_m$ is an even integer and thus $\gamma_n = 0$. When $0 \leq n \leq m$, $|\gamma_n| \leq 4^n$. Since $|\gamma_m| = 4^m$, we see that

$$\frac{|f(x+\delta_m) - f(x)|}{|\delta_m|} = |\sum_{n=0}^{m} \frac{3^n}{4^n}\gamma_n| \geq 3^m - \sum_{n=0}^{m-1} 3^n = \frac{1}{2}(3^m + 1). \tag{1}$$

Therefore since $\delta_m \to 0$ as $m \to \infty$ we conclude that f is not differentiable at x. □

2 Equicontinuity and Arzela-Ascoli

Suppose we want to find the root of a polynomial on a computer. Normally what one does is to use some iterative technique (like Newton's method) whereby you have an initial "guess" x_0, then you improve it to x_1, improve that to x_2 and so on. You hope that this sequence will converge to the correct answer. How does one prove these sorts of things ? Well, at least if you prove that this sequence is bounded, then there is some hope because every bounded sequence of real numbers has a convergent subsequence. So at least a subsequence converges.

If you want to play the same game with solving a differential equation on a computer, then naively, you will take a "guess" $f_0(x)$, then improve it to $f_1(x)$ and so on. If you at least want a convergent subsequence, then is proving that f_k are "bounded" enough ? What does "bounded" mean for a family of functions in the first place anyway ?

We say that f_n is pointwise bounded if for every $x \in E$ the sequence of numbers $f_n(x)$ is bounded, i.e. there is a finite function $\phi(x)$ such that $|f_n(x)| \leq \phi(x)$.

We say that f_n is uniformly bounded if $|f_n(x)| < M$ for all n and all $x \in E$. If f_n converges uniformly to some function $f(x)$ then $|f_n(x)| \leq \epsilon + |f(x)|\ \forall\ n \geq N$ and all $x \in E$. This means that $f_n(x)$ is definitely pointwise bounded. Suppose all the f_n are bounded functions, i.e., $|f_n(x)| < M_n$ which is independent of x, then of course $|f(x)| \leq M_N + \epsilon$ and hence f_n are actually uniformly bounded.

Unfortunately, even if f_n are uniformly bounded sequence of continuous functions on E, there need not exist a subsequence which converges even pointwise on E! This is not true even if E is compact.
Here is a counterexample : Take $f_n(x) = \sin(nx)$ on $[0, 2\pi]$. This is a uniformly bounded family. Suppose a subsequence $f_{n_k}(x) = \sin(n_k x)$ converged pointwise to some $f(x)$, then $\lim_{k\to\infty}(\sin(n_k x) - \sin(n_{k+1}x))^2 = 0$ on $[0, 2\pi]$. An easy calculation shows that $\displaystyle\int_0^{2\pi} (\sin(n_k x) - \sin(n_{k+1}x))^2 = 2\pi$. However, if you could interchange limits and integrals, you will see that $\lim_{k\to\infty} \int_0^{2\pi}(\sin(n_k x) - \sin(n_{k+1}x))^2 dx = 0$. The fact that you can do so follows from a deep theorem in measure theory. It is not easy to prove such things with the machinery we have developed so far. But anyway this is just a counterexample, so we don't need to worry too much about.

Another natural question is "Even if you have a convergent sequence, is there a uniformly convergent subsequence if the sequence is uniformly bounded on a compact set?" The answer is still unfortunately, no :
Take $f_n(x) = \frac{x^2}{x^2+(1-nx)^2}$ on $[0, 1]$. This family of functions is of course uniformly bounded by 1. It converges pointwise to $f(x) = 0$. However, $f_n(1/n) = 1$ which means that no subsequence can converge uniformly to 0 (If it did, then $|f_{n_k}(x)| < \epsilon < 1$ for all x and $k > N$. But $f_{n_k}(1/n_k) = 1 > \epsilon$).

We need another condition called equicontinuity. A family $\mathcal{F}$ of complex-valued functions defined on a set E in a metric space X is said to be equicontinuous on E if for every $\epsilon > 0$ there exists a $\delta_\epsilon > 0$ such that $|f(x) - f(y)| < \epsilon$ whenever $d(x, y) < \delta_\epsilon$ for all $f \in \mathcal{F}$, i.e., δ_ϵ depends only on ϵ and not on x, y, f. In particular an equicontinuous family consists of uniformly continuous functions. It will turn out later on that this is the missing condition.