Notes for 24 Mar (Friday)

1 Recap

1. Constructed an example of a function that is everywhere continuous but nowhere differentiable.

2. Defined equicontinuity as possible hypothesis to make in order to study compact subsets of $\mathcal{C}(X)$.

2 Equicontinuity

Here are examples and counterexamples of equicontinuous families of functions :

1. The previously seen family $f_n(x) = \frac{x^2}{x^2+(1-nx)^2}$ on $[0,1]$ is NOT equicontinuous. Indeed, if $|f_n(x) - f_n(y)| < \epsilon < 1$ for $|x-y| < \delta_\epsilon$, then take n so large that $\frac{1}{n} < \delta_\epsilon$. For such an n, $|f_n(1/n) - f_n(0)| = 1 > \epsilon$.

2. The family $f_n(x) = \frac{\sin(nx)}{n^2}$ is equicontinuous on all of $\mathbb{R}$. Indeed, $|f_n(x) - f_n(y)| = \frac{1}{n^2}|(\sin(nx) - \sin(ny))| = \frac{1}{n^2}|\displaystyle\int_x^y n\cos(nt)dt| \leq \frac{1}{n}|y-x| < \epsilon$ when $|y-x| < \epsilon$.

3. The family $f_n(x) = \frac{e^{-nx}}{n^2}$ is equicontinuous on $[0,1]$ but not so on all of $\mathbb{R}$. Indeed, on $[0,1]$, $|f_n(x) - f_n(y)| = \frac{1}{n^2}\displaystyle\int_x^y ne^{-nt}dt \leq \frac{1}{n}|x-y| < \epsilon$ when $|x-y| < \epsilon$. However, on all of $\mathbb{R}$ if $|f_n(x) - f_n(y)| < \epsilon$ when $|x-y| < \delta$, then assume $y = 0$ and $x = -\frac{\delta}{2}$. This means that $|\frac{e^{n\delta/2}-1}{n^2}| < \epsilon$. But as $n \to \infty$ we get a contradiction.

So the moral of the story is that equicontinuity is sort of (but not exactly, for god's sake not exactly) like saying that $|f_n'(x)|$ is uniformly bounded independent of x and n. For now we prove two useful theorems.

Theorem 1. *If f_n is a pointwise bounded sequence of complex-valued functions on a countable set E, then f_n has a subsequence f_{n_k} such that $f_{n_k}(x)$ converges to some $f(x)$ for every $x \in E$.*

Proof. Suppose the elements of E are enumerated as $x_1, x_2, \ldots$. Now $f_n(x_1)$ is bounded independent of n. Thus it has a convergent subsequence $f_{n_k}(x_1)$. In other words, for x_1, you have a subsequence of the f_n that converges. Take this subsequence for x_2, namely,

$f_{n_k}(x_2)$. This is once again bounded by assumption (independent of k). Thus again by Bolzano-Weierstrass, it has a further subsequence $f_{n_{k_l}}(x_2)$ that converges. You can continue like this for every x_i. In other words, for x_1 you have a convergent subsequence $f_{1,1}, f_{1,2}, \ldots$. For x_2 you have a further convergent subsequence of the previous one which we shall call as $f_{2,1}, f_{2,2} \ldots$ and so on. Consider the diagonal $f_{1,1}, f_{2,2}, \ldots$. This is definitely a subsequence (except for the first few terms) of all the subsequences. Therefore it converges for every x_i. ☐

The second theorem shows why equicontinuity might be necessary.

Theorem 2. *If K is a compact set such that f_n are continuous functions on K, and if f_n converges uniformly to some function f on K, then f_n are equicontinuous on K.*

Proof. Firstly, since K is compact, all the f_n are uniformly continuous. Secondly, since f_n are uniformly convergent, they are uniformly Cauchy. This means that for every $\epsilon > 0$ there exists an N (depending on ϵ) such that $n, m \geq N$ implies that $|f_n(x) - f_m(x)| < \epsilon$. Now what does it mean for these guys to be equicontinuous ? It means that $|f_n(x) - f_n(y)| < \epsilon$ whenever $d(x, y) < \delta_\epsilon$ where δ_ϵ does not depend on x, y or even n. We can find $\delta_{\epsilon,n}$ for every n by uniform continuity but the problem is that it depends on n.
So, $|f_n(x) - f_n(y)| \leq |f_n(x) - f_N(x)| + |f_N(x) - f_N(y)| + |f_N(y) - f_n(y)| < \epsilon + \epsilon + \epsilon = 3\epsilon$ when $n \geq N$ and $d(x, y) < \delta_{\epsilon,N}$. So if $\delta = \min(\delta_{\epsilon,1}, \delta_{\epsilon,2} \ldots \delta_{\epsilon,N})$ then we are done. ☐

Finally we prove the celebrated Arzela-Ascoli theorem.

Theorem 3. *If K is a compact subset of $\mathbb{R}$, f_n are continuous on K, and if f_n is pointwise bounded and equicontinuous on K, then*

1. *f_n are uniformly bounded on K, and*

2. *f_n contains a subsequence f_{n_k} that converges uniformly to some continuous function f.*

Proof.

1. Since f_n are continuous on a compact set K, they are all bounded, i.e., $|f_n(x)| < M_n$. By assumption, they are also pointwise bounded, i.e., $|f_n(x)| < \phi(x)$. We want $|f_n(x)| < M$. Fix a y. Since f_n are equicontinuous, there is a δ such that $d(x, y) < \delta \Rightarrow |f_n(x) - f_n(y)| < \epsilon$. Thus $|f_n(x)| < |f_n(y)| + \epsilon < \phi(y) + \epsilon$. Now the open sets $U_y = B_{\delta/2}(y)$ cover K. Since K is compact, only finitely many are necessary. Call them $U_{y_1}, U_{y_2}, \ldots U_{y_n}$. Let $M = \epsilon + max(\phi(y_1), \phi(y_2), \ldots, \phi(y_n))$. Since any $x \in K$ is in one of the balls, let's say it is in U_{y_1}. Then $|f_n(x) - f_n(y_1)| < \epsilon$ because $d(x, y_1) < \delta$. Hence $|f_n(x)| < |f_n(y_1)| + \epsilon < M$.

2. Since the rationals are countable, we can extract a subsequence f_{n_k} (owing to one of the previous theorems we proved) such that f_{n_k} converges pointwise to some function f on the rationals. We shall prove that the very same subsequence actually converges uniformly on all of $\mathbb{R}$. Let's simplify notation and call f_{n_k} as g_k. The idea is that rationals are dense, i.e., every real number is close to some rational. So it seems reasonable that because the $g_k = f_{n_k}$ converge for that rational and they are equicontinuous, they ought to for the real number in consideration as well.
   Fix an $x \in K$. Now $|g_i(x) - g_j(x)| \leq |g_i(x) - g_i(x_0)| + |g_i(x_0) - g_j(x_0)| + |g_j(x_0) - g_j(x)|$

when x_0 is a rational chosen to be δ-close to x such that $|g_i(x) - g_i(x_0)| < \epsilon$ for all i and all x (by equicontinuity, such a δ exists). Choose $i, j > N$ such that $|g_i(x_0) - g_j(x_0)| < \epsilon$. This can be done because g_k converge on rationals. So consider the open sets $U_q = B_\delta(q)$ where q is a rational. Only finitely many of these U_{q_i} are necessary to cover K. Therefore, choose $N = \max(N_{q_1}, N_{q_2}, \ldots)$. If $i, j > N$, indeed $|g_i(x) - g_j(x)| < 3\epsilon \ \forall \ x \in K$. This shows that indeed g_i are uniformly Cauchy and hence uniformly converge.

$\square$

Note that this proof applies almost word-to-word for K being a compact subset of a metric space. It is just that we need to know that there is a countable dense subset for every compact metric space. For the reals, we know that rationals are dense and are countable. For an arbitrary compact metric space K, note that you need only finitely many balls of size $\frac{1}{n}$ (where n ranges over all positive integers) to cover K. (Indeed, $B_{1/n}(p)$ where p ranges over all K covers K. Since K is compact you need only finitely many.) Since a countable union of countable sets is countable, the centres of these countably many balls form a dense subset.