Notes for 25th Jan (Wednesday)

1 Recap

1. Defined open sets, closed sets, dense subsets, perfect sets (every point is a limit point), closure, and boundary.

2. Proved that a set is closed if and only if it contains all its limit points, proved that $E \cup E' = \bar{E}$, and finite intersections and arbitrary unions of open sets is open (and likewise for closed).

3. Defined the notion of relatively open w.r.t to Y (i.e. every point p is contained in a set $Y \cap B_r(p)$ which is completely contained in E).

2 Metric spaces (cont'd...)

Suppose (X, d) is a metric space. Suppose $Y \subset X$. A subset E of Y is said to be open relative to Y if and only if $E = Y \cap G$ where G is an open subset of X.
Pf : Suppose $E = Y \cap G$. Then, given $p \in E$, since $p \in G$, there is a neighbourhood $B_r(p) \subset G$. Now the set $B_r(p) \cap Y \subset E$ is a neighbourhood of p in Y. Therefore E is open relative to Y.
If E is open relative to Y, then every point $p \in E$ has a neighbourhood in Y, i.e., $B_r(p) \cap Y \subset E$. Now let $G = \cup_{p \in E} B_r(p)$. This is an open set. Also, $E = G \cap Y$. (Why?)

3 Compactness

We noticed that even perfectly nice, continuous functions can fail to have maxima/minima over certain sets. For instance, $\frac{1}{x}$ on $(0, 1)$. There are two problems with this function. Firstly, we can force it to have a minimum by considering it over $(0, 1]$. However, we cannot make it have a maximum. What is going wrong is that the set on which it is defined is not closed. (So you can go arbitrarily close to the maximum without ever reaching it.) If you think closedness is the only problem, think again - Consider the perfectly lovely $f(x) = x$ on $\mathbb{R}$. This does not have extrema because it grows without any limit. (Much like stupidity of us humans.) So it seems that continuous functions on closed sets that are bounded ought to have extrema. We will prove that indeed this is the case. In fact, we will prove something more general. For this we need a definition of certain sets whose special cases are "Closed bounded sets". The name of such a set is "compact".

An open cover $\{G_\alpha\} \subset \mathcal{P}(X)$ of E is a collection of open sets G_α such that $E \subset \cup_\alpha G_\alpha$. A compact set $E \subset X$ where (X, d) is a metric space is one for which every open cover $\{G_\alpha\}$ has a finite subcover, i.e., there exists a *finite* subcollection $G_1, \ldots, G_n$ such that $E \subset \cup_{i=1}^n G_i$.

It seems like a random definition. It seems to be quite hard to find examples. (Well, every finite set is compact, but that is a triviality.) Before we come up with examples, let's first prove that compactness behaves well under relative-openness/closedness :

Suppose $K \subset Y \subset X$. Then K is compact relative to Y if and only if it is compact relative to X.
Pf: If K is compact relative to Y, then given any cover of relatively open sets $\tilde{G}_\alpha \subset Y$ then there exists a finite subcover of relatively open sets $\tilde{G}_1, \ldots, \tilde{G}_n$. Now given an open cover (relative to X) G_α, consider the relatively open cover $\tilde{G}_\alpha = G_\alpha \cap Y$. By compactness, $K \subset \cup_i \tilde{G}_i = \cup_i (G_i \cap Y) = Y \cap \cup_i G_i \subset \cup_i G_i$. Thus it is compact relative to X.
If K is compact relative to X, then we shall prove that it is compact relative to Y. Indeed, given any cover of relatively open sets $\tilde{G}_\alpha \subset Y$, we know that $\tilde{G}_\alpha = G_\alpha \cap Y$ for some G_α that are open in X. Therefore G_α form a cover. Hence, there is a finite subcover G_i. This means that $K \subset \cup_i G_i$. But $K \subset Y$. Hence $K \subset Y \cap \cup_i G_i = \cup_i \tilde{G}_i$.

Because of the above theorem, it makes sense to talk of compact metric spaces (without referring to whether the metric space is a subspace of a larger space).

Here is a pleasant property of compactness (that confirms the usefulness of the definition) :
Compact subsets of metric spaces are closed.
Pf : If $K \subset X$ is compact, then suppose $x \in K^c$. We shall prove that x has a neighbourhood contained in K^c by studying the "minimum" distance between x and points of K. Indeed, if $p \in K$ consider the neighbourhood $N_p = B_{d(p,x)/2}(p)$. When p ranges over all points in K, these neighbourhoods form an open cover of K. Since K is compact, there exists a finite subcover $N_{p_1}, N_{p_2}, \ldots N_{p_n}$. Let $r = min(d(p_i, x)$. Then $B_r(x)$ is disjoint from N_{p_i} for all $1 \leq i \leq n$. Therefore $B_r(x) \subset K^c$. Thus K is closed.

Here is another such property :
Closed subsets of compact sets are compact.
Pf: If $C \subset K \subset X$ then if $\{U_\alpha\}$ is an open cover of C, then $\{U_\alpha\} \cup \{C^c\}$ is an open cover of K. (Here is where we need to use the closedness of C. However, we only know that C is closed relative to K, i.e., (it can be proven as an exercise) that $C = A \cap K$ where A is closed in X. But compact sets are closed and hence so is C closed in X.) Since K is compact, there exists a finite subcover of it given by $U_1, U_2, \ldots U_n, C^c$. Thus $U_1, \ldots U_n$ cover C.

As a corollary, we see that the intersection of closed and compact is compact.

Yet another :
Compact subsets of metric spaces are bounded.
Pf: Let $p \in K$. Consider the open cover $B_r(p)$ where r ranges over all strictly positive

reals. Since K is compact there exists a finite subcover $B_{r_1}(p), \ldots$. Therefore the maximum radius R of the r_i exists. This easily implies the result.

Our aim now is to finally prove the Heine-Borel theorem -

Theorem 1. *Closed bounded subsets of* $\mathbb{R}^n$ *are compact (and vice-versa).*

By the previous theorems, the "vice-versa" part is done. Firstly, we claim that it is enough to prove that closed rectangles are compact. Indeed, given a closed bounded subset, it is contained in a sufficiently large closed rectangle (Why?) and we proved that closed subsets of compacts sets are compact.

So, we need to prove that

Theorem 2. *Closed rectangles* $R = [a_1, b_1] \times \ldots$ *in* $\mathbb{R}^n$ *are compact.*

What does this entail ? Given an open cover U_α of R, we need to prove that it has a finite subcover U_i, i.e., $R \subset U_1 \cup U_2 \ldots U_n$.

As usual, suppose not. We will derive a contradiction somehow. Let $c_j = \frac{a_j+b_j}{2}$. Since R does not have a finite subcover, at least one rectangle I_1 out of the 2^n rectangles formed by $[a_j, c_j]$ and $[c_j, b_j]$ is not covered by finitely many U_α. Now subdivide I_1, rinse and repeat.

Thus we will have obtained a collection of closed rectangles

$$I_1 \supset I_2 \supset I_3 \ldots \tag{1}$$

such that I_n is not covered by a finite subcollection of U_α and I_n is "small", i.e., $\|\vec{x} - \vec{y}\| \leq \frac{\delta}{2^n}$ where $\delta = \|\vec{a} - \vec{b}\|$ for all $\vec{x}, \vec{y} \in I_n$.

The basic idea is that *if* $\cap_{n=1}^{\infty} I_n$ is not empty and hence contains a point p (sounds believable), then since $p \in U_\beta$ for some β, there is a neighbourhood $B_r(p) \subset U_\beta$. But I_n are getting smaller and smaller (and they all contain p). So at least one of them has to be in this neighbourhood (which one ? You need to use the Archimedian property of reals here). But that is a contradiction because it is covered by U_β.

So we have reduced our problem to proving the following lemma.

Lemma 3.1. *If* $I_1 \supset I_2 \ldots$ *is a collection of closed rectangles in* $\mathbb{R}^m$*, then* $\cap_n I_n$ *is not empty.*

Proof. Firstly, we will prove this for closed intervals in $\mathbb{R}$. If indeed, $[a_1, b_1] \supset [a_2, b_2] \ldots$, then consider the set E consisting of all a_n. It is of course non-empty and bounded above because $a_1 \leq a_n \leq b_n \leq b_1$. Therefore by the least upper bound property of reals, $x = \sup E$ exists. We claim that indeed $x \in [a_k, b_k]\ \forall\ k$. Indeed, $a_k \leq x\ \forall\ k$. Since $a_n \leq a_{n+m} \leq b_{n+m} \leq b_n$ it is clear that $x \leq b_n\ \forall\ n$.
Given this, if $I_1 = [a_{11}, b_{11}] \times [a_{12}, b_{12}] \ldots \supset I_2 = [a_{21}, b_{21}] \times [a_{22}, b_{22}] \ldots$, then since $[a_{i1}, b_{i1}] \supset [a_{i2}, b_{i2}] \supset \ldots\ \forall\ 1 \leq i \leq m$, we see that there exist $x_i \in \cap_{n=1}^{\infty} [a_{in}, b_{in}]$ by the one-dimensional version proved above. Now the point $(x_1, x_2, \ldots, x_m)$ is in all I_n. □