Notes for 27th Jan (Friday)

1 Recap

1. We characterised relative openness w.r.t Y as subsets of the form $U \cap Y$ where U is a "usual" open subset of X.

2. We defined compactness, proved that compact subsets are closed and bounded, closed subsets of compact sets are compact, and the Heine-Borel theorem (closed bounded subsets of compact sets are compact).

2 Compactness (cont'd)..

Before proceeding further, we note that the property of intersection of rectangles we used in the proof of Heine-Borel is actually common to compact sets in general metric spaces.

Lemma 2.1. *If $\{K_\alpha\}$ is an arbitrary collection of compact subsets of a metric space X such that the every finite intersection is nonempty, then $\cap K_\alpha$ is also nonempty.*

Proof. Suppose $\cap K_\alpha = \phi$. Then $\cup K_\alpha^c = X$. This means that the open sets $U_\alpha = K_\alpha^c$ cover X and hence cover $K_\beta\ \forall\ \beta$. But each K_β is compact and hence need only finitely many of the U_α to cover it. Thus $K_1 \subset K_{\alpha_1}^c \cup K_{\alpha_2}^c \ldots K_{\alpha_n}^c$. This means that $K_1 \cap K_{\alpha_1} \cap K_{\alpha_2} \ldots = \phi$ which is a contradiction. □

The Heine-Borel theorem implies the following.

Theorem 1. *Let $E \subset \mathbb{R}^m$. The following are equivalent.*

1. *E is compact.*

2. *E is closed and bounded.*

3. *Every infinite subset of E has a limit point in E.*

Later on when we study sequences, we will easily see that this is also the same as saying that every sequence in E has a convergent subsequence (whose limit lies in E).

Proof. The first two are equivalent by the Heine-Borel theorem. If A is an infinite subset of E, then A contains a countably infinite subset $x_1, x_2, \ldots$.
Suppose E is compact. If A has no limit point in E, then every point $p \in E$ has a neighbourhood $N_p = B_{r_p}(p)$ such that it does not contain any point of A (other than p itself possibly). Such neighbourhoods form an open cover of E. Therefore, finitely many such neighbourhoods $N_{p_1}, N_{p_2}, \ldots N_{p_k}$ cover E and hence A. This is a contradiction.
Suppose every such A has a limit point in E, we will prove that E is

1. Bounded : If not, then there exists a point $u_R \in E$ for every $R > 0$ such that $\|u_R\| > R$. Thus $\{u_R\}$ cannot have a limit point (why?) but it is an infinite subset of E.

2. Closed : We will prove that E contains all of its limit points. Indeed, if p is a limit point of E, then for every integer $j > 0$ every neighbourhood $B_{1/j}(p)$ contains a point $q_j \neq p$ from E. The set A of all such q_j has a limit point in E. We claim that it has only one limit point, namely, p. Thus $p \in E$.
   Indeed, if $p \neq q$ is a limit point of $\{q_j\}$, then the neighbourhood $B_{d(p,q)/2}(q)$ has only finitely many q_j. This is a contradiction.

□

Please note that while compact sets in general metric spaces are closed and bounded, (and that infinite subsets having limit points in E is still equivalent to being compact), not every closed and bounded set is compact. (Take all rationals satisfying $2 < p^2 < 3$.)

Theorem 2. *(Weierstrass) : Every bounded infinite subsets of $\mathbb{R}^m$ has a limit point in $\mathbb{R}^m$.*

Proof. Indeed, such a set is contained in a large closed rectangle (which is compact by Heine-Borel). Thus it has a limit point in that rectangle. □

3 Perfect sets

Theorem 3. *Every nonempty perfect set P in $\mathbb{R}^k$ is uncountable.*

Proof. Suppose P is countable, and its elements are given as $x_1, x_2, \ldots$. Since P is perfect, each of these is a limit point of P. Let V_1 be any open neighbourhood of x_1. We will construct a collection of open sets V_i such that $\bar{V}_1 \supset V_1 \supset \bar{V}_2 \supset V_2 \supset \bar{V}_3 \ldots$ such that $x_n \notin \bar{V}_{n+1}$ and $V_n \cap P$ is not empty. Let $K_n = \bar{V}_n \cap P$. By one of the lemmas we proved earlier, $\cap_n K_n$ is not empty. But $x_n \notin K_{n+1}$ and therefore $\cap_n K_n$ has to be empty. This is a contradiction.

One can construct the neighbourhoods V_n inductively using the fact that every point of P is a limit point of P. Indeed, suppose $V_1, V_2, \ldots, V_n$ have been constructed. Suppose $x_n \in V_n$. (Otherwise, we are done because V_{n+1} can be chosen to be a smaller neighbourhood of any point of $V_n \cap P$.) Then since x_n is a limit point of P, V_n contains infinitely many points from P. Just choose V_{n+1} to be any small neighbourhood of any point $\neq x_n$ in $V_n \cap P$ that does not contain x_n and is completely contained within V_n. □

It seems that perfect sets must always contain an open interval since they seem to consists exclusively of limit points (as opposed to isolated points). But this is not true !! Here is a famous counterexample called the Cantor set. This is closely connected to fractals and other fancy stuff.

Take $[0, 1]$. Throw out the middle third $(1/3, 2/3)$. You are left with $[0, 1/3] \cup [2/3, 1]$. (That is, in the ternary expansion, the first digit after the decimal point is either 0 or 2,

or if it is 1, then all the digits after that are 0.) Now do the same thing to the remaining intervals. (Throw out the middle third.) Rinse and repeat ad infinitum. The resulting object has either only 0 or 2 in its ternary expansion, or if it has a 1 then it only has zeroes after the 1. So it is definitely uncountable. (It has only binary sequences.) It definitely does not have an open interval (because it virtually does not have more than a single 1 in its ternary expansion).
It is a perfect set too. Indeed, it is the intersection of countably many compact sets, and is hence closed. It is also bounded (obviously). Therefore it is compact. (It is non-empty even from general considerations of compactness.) We have to show that it has no isolated points. This is clear from the ternary representation.

The weird thing is tha if you throw a dart at $[0, 1]$, you are almost surely not going to hit the Cantor set despite it having uncountably many elements ! (This is closely related (but not exactly because of) to the observation that it has no open intervals.)

4 Connected sets

Suppose $E \subset X$ where (X, d) is a metric space. E is said to be *connected* if it is NOT equal to $(U_1 \cap E) \cup (U_2 \cap E)$ where U_1, U_2 are open subsets of X and $(U_1 \cap E) \cap (U_2 \cap E) = \phi$. In other words, it cannot be written as a disjoint union of two open sets (open relative to E). This sounds like a ****ed up definition of an intuitive, nice concept involving a simple English word "connected". But it is quite useful.

First order of business :

Theorem 4. *If $E \subset \mathbb{R}$, then E is connected if and only if, whenver $x, y \in E$, then all z such that $x < z < y$ also belongs to E.*

Proof. Suppose E is connected. If there is a z in (x, y) such that $z \notin E$, then $E = ((-\infty, z) \cap E) \cup ((z, \infty) \cap E)$. This is a contradiction.
Suppose E is not connected, i.e., $E = (U_1 \cap E) \cup (U_2 \cap E)$. Suppose $x \in U_1 \cap E$ and $y \in U_2 \cap E$ such that $x < y$. Let U be the set of all $x \leq t \in U_1 \cap E$. Clearly U is bounded above by y and is nonempty. By the least upper bound property, there exists $z_1 = \sup U$. Likewise, there exists a $z_2 =\in V$ where V consists of all $y \geq t \in U_2 \cap E$. There are only two possibilities :

1. $z_1 < z_2$. Clearly any z in (z_1, z_2) does not lie in E.

2. $z = z_1 = z_2$. Suppose $z \in E$. Without loss of generality assume that $z \in U_1 \cap E$. Then $(z - \epsilon, z + \epsilon) \in U_1$ for all sufficiently small $\epsilon > 0$. Since z is a limit point of $U_2 \cap E$, such a neighbourhood contains points from $U_2 \cap E$. But this means that $U_1 \cap U_2 \cap E = U_1 \cap E \cap U_2 \cap E \neq \phi$. A contradiction.

□

We will see connectedness again later on. But all of this is a part of topology. (What is fondly known as point-set topology because it involves little more than set theory.)