Notes for 2 Feb (Thursday)

1 Recap

1. Defined sequences and subsequences. Proved basic properties of sequences.
2. Defined Cauchy sequences, completeness, and proved that $\mathbb{R}^k$ is complete.

2 Sequences (cont'd..)

Here is a familiar definition : A sequence of reals s_n is said to be monotonically increasing if $s_n \leq s_{n+1} \; \forall \; n$. Likewise for monotonically decreasing. If it is either, then it is said to be monotonic. (Like this lecture of mine.)
Here is a theorem :

Theorem 1. *Suppose $\{s_n\}$ is monotonic. Then $\{s_n\}$ converges if and only if it is bounded.*

Proof. If it converges, of course it is bounded. (Why?)
If it is bounded and monotonically increasing (without loss of generality), then let $s = \sup\{s_n\}$. I claim that $s_n \to s$. Indeed, given $\epsilon > 0$, since s is the supremum, $s - \epsilon$ is no longer an upper bound. Therefore, there exists an N so that $s_N > s - \epsilon \geq s_N - \epsilon$ (which means that $s_n \geq s_N > s - \epsilon \geq s_n - \epsilon \; \forall \; n > N$. This means that $|s_n - s| < \epsilon \; \forall \; n \geq N$. $\square$

3 Upper and lower limits

Firstly, a small definition : We say that $x_n \to \infty$ if for every $M > 0$, there exists a natural N_M such that $x_n \geq M \; \forall \; n > N_M$. Likewise for $-\infty$.
From now onwards, we will allow the symbols $+\infty$ (simply written as ∞) and $-\infty$ in our "number system". This is called "Extended Real numbers". These symbols satisfy the following properties :

1. $-\infty < x < \infty$ for all real numbers x. (So if a set E is not bounded from above, then $\sup E = \infty$.)
2. If $x \in \mathbb{R}$ then $x + \infty = \infty$, $x - \infty = -\infty$, $\frac{x}{\infty} = \frac{x}{-\infty} = 0$.
3. If $x > 0$, then $x.\infty = \infty$ and $x.(-\infty) = -\infty$.
4. Likewise for $x < 0$.
5. $\infty.\infty = \infty$, $\infty.(-\infty) = (-\infty).\infty = -\infty$, $\infty + \infty = \infty$ and $-\infty - \infty = -\infty$.

Note that the extended reals do not form a field.

Suppose x_n is a sequence of real numbers. Let E (in the extended reals) be the set of all subsequential limits of $\{x_n\}$ (possibly including ∞, $-\infty$ if necessary).

Define s^* (written as $\limsup x_n$ and read as "upper limit" or "limit superior") as $s^* = \sup E$ (which is allowed to be ∞) and likewise $s_* = \inf E$ (allowed to be $-\infty$). Note that whether the sequence converges to a limit or not, the lim sup and lim inf always exist. (Why ? If the sequence is bounded then it contains a convergent subsequence. So $E \neq \phi$ and thus it makes sense to talk of lim sup and lim inf. If the sequence is unbounded, then a subsequence converges to ∞ or $-\infty$. Therefore E is still not empty.)

Here is a very important property of lim sup and lim inf :

Theorem 2. 1. *s^* and s_* are in E, i.e., there exist subsequences converging to the lim sup and lim inf. In other words, the supremum of all subsequential limits is in fact the* maximum *of all subsequential limits (and likewise).*

2. *If $x > s^*$ there is an integer N such that $n > N$ implies that $x_n < x$.(Likewise for lim inf.) Moreover, s^* and s_* are the only two numbers satisfying these properties.*

Proof. We will only prove for s^*. (The case for s_* follows by taking negatives.)

1. Since s^* is the supremum of E, $x_\epsilon - \frac{\epsilon}{2} < s^* - \frac{\epsilon}{2} < x_\epsilon$ where x_ϵ is a subsequential limit $x_\epsilon = \lim_{k\to\infty} x_{n_k}$. This means that for $k \geq N_\epsilon$, $|x_{n_k} - x_\epsilon| < \frac{\epsilon}{2}$ thus implying that $|s^* - x_{n_{N_\epsilon}}| < \epsilon$. Therefore $x_{n_{N_{1/k}}} \to s^*$ as $k \to \infty$.

2. Suppose not. That is, there exists $x > s^*$ such that for every N there exists an $x_{n_N} \geq x$. Then clearly the lim sup of the subsequence x_{n_k} is $\geq x > s^*$. This is a contradiction because the lim sup of the subsequence is a subsequential limit.

Suppose $s_1 ands_2$ are two numbers satisfying the above properties (for s^*). W.Lo.G $s_1 > s_2$. This means that for all $n > N$ we have $x_n < s_1 + \frac{s_1-s_2}{2}$. But s_1 is apparently a subsequential limit. This is a contradiction. □

Sometimes people define lim inf and lim sup in the following way (which is a theorem in our definition). This is actually very useful to *calculate* the lim sup and lim inf.

Theorem 3.

$$\limsup = \lim_{N\to\infty} \sup_{n\geq N} x_n$$

$$\liminf = \lim_{N\to\infty} \inf_{n\geq N} x_n$$

Proof. As usual we will only prove the theorem for lim sup. Indeed $b_N = \sup_{n\geq N} x_n$ exists as a sequence of extended real numbers. Notice that $b_{N+1} \leq b_N$. Thus the limit (call it y) exists as an extended real number (because b_N is monotonically decreasing). We will prove that this limit satisfies both properties that we mentioned in the previous theorem.

1. y is a subsequential limit : Given $\epsilon > 0$, choose N to be so large that $0 < b_{N_\epsilon} - y < \epsilon$. Now choose n to be so large that $b_{N_\epsilon} > x_{n_\epsilon} > b_{N_\epsilon} - \epsilon$. Thus $|x_{n_\epsilon} - y| < \epsilon$. Therefore the subsequence $x_{n_{1/k}}$ converges to y as $k \to \infty$.

2. If $x > y$ then $x_n < x$ for $n > N$: If $x > y$ then for $N > N_1$ surely $x > b_N$. This means that $x > x_n$ for all $n \geq N > N_1$.

□

Also,

Theorem 4. *1. If* $s_n \leq t_n$ *for* $n \geq N$ *then* $\limsup s_n \leq \limsup t_n$ *and* $\liminf s_n \leq \liminf t_n$.

2. $\lim s_n = s \Leftrightarrow \limsup s_n = \liminf s_n = s$.

Proof. 1. Exercise. (Just use our original definition.)

2. Of course if the maximum and minimum of subsequential limits coincide then all subsequences converge to the same limit and hence we are done.

□

Here are some examples :

1. For the sequence $1, -1, 1, -1, \ldots$, $\limsup = 1$ and $\liminf = -1$.

2. For a sequence consisting of all rationals (which we know are countable), the lim sup is $\limsup = \infty$ and $\liminf = -\infty$. Also, every real is a subsequential limit. (Proof by contradiction.)

4 Special sequences

The so-called Sandwich observation/rule is quite useful : If $0 \leq x_n \leq s_n$ and $\lim_{n\to\infty} s_n = 0$ then $\lim_{n\to\infty} x_n = 0$. These sequences occur frequently.

1. If $p > 0$, then $\lim_{n\to\infty} \frac{1}{n^p} = 0$.
   Pf : If we want $\frac{1}{n^p} < \epsilon$ we simply need to choose $n > (1/\epsilon)^{1/p}$.

2. If $p > 0$, then $\lim_{n\to\infty} p^{1/n} = 1$.
   Pf : If $p = 1$ the result is trivial. By taking reciprocals it follows that we only need to consider $p > 1$. Of course $p^{1/n} = 1 + x_n$ for some $x_n > 0$. Thus $(1 + x_n)^n = p$. Therefore $1 + nx_n < p$. This means that $x_n \to 0$.

3. $\lim_{n\to\infty} n^{1/n} = 1$.
   Pf : Note that $n^{1/n} > 1$ for $n \geq 2$. Thus $n^{1/n} = 1 + y_n$ where $y_n > 0$. Thus $(1 + y_n)^n = n \Rightarrow n(n-1)y_n/2 < n$. This means that $y_n \to 0$.

4. If $p > 0$ and $\alpha \in \mathbb{R}$ then $\lim_{n\to\infty} \frac{n^\alpha}{(1+p)^n} = 0$.
   Pf : Note that $(1+p)^n > C(n,k)p^k$. Suppose $\alpha + 2 > k > \alpha + 1$. Then

$$(1+p)^n > C_\alpha (n-\alpha)^{\alpha+1} > C\frac{n^{\alpha+1}}{2^{\alpha+1}} \tag{1}$$

   This shows the result trivially.

5. If $|x| < 1$ then $\lim_{n\to\infty} |x|^n = 0$.
   Pf : Put $\alpha = 0$.

5 Series

Given a sequence $\{a_n\}$, a series is naively speaking, the sum of all terms of the sequence. More rigorously, define the *sequence* of partial sums $s_n = \sum_{k=1}^{n} a_k$. We say that the series converges to L if and only if the partial sums $s_n \to L$. Otherwise we say that the series diverges. This definition itself sheds light on the so-called Zeno paradox - Achilles allows a head start of 10m to the tortoise (who runs at half his speed, that is, at $1m/s$). After $5s$ he reaches the starting point of the tortoise. After 2.5 more seconds he reaches where the tortoise was at $t = 5s$ and so on. This "means" he can never overtake the tortoise.

The fact that real sequences converge if and only if they are Cauchy can be translated into : A series $\sum a_n$ converges if and only if, given an $\epsilon > 0$, there exists a natural N_ϵ such that $n, m > N_\epsilon$ implies that $|\sum_{k=n}^{m} a_k| < \epsilon \ \forall\ n, m > N_\epsilon$. In particular,

Lemma 5.1. *The divergence test If a series $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} |a_n| = 0$.*

Of course, just because $a_n \to 0$ does NOT mean that $\sum a_n$ converges. The classic example is the Harmonic series : $\sum \frac{1}{n}$ which diverges very slowly (logarithmically). But $1/n \to 0$. There are several ways to prove this. (The easiest if you already know integral calculus is to bound it from below by the integral of $\frac{1}{x}$. But since we don't assume knowledge of calculus yet, we will prove it later.)

The monotonicity theorem has a counterpart for series : A series of non-negative terms converges if and only if it is bounded.

Here is a useful little result :

Lemma 5.2. *If $\sum_{k=1}^{\infty} |a_k|$ converges, then so does $\sum_{k=1}^{\infty} a_k$.*

Proof. Indeed, $|\sum_{k=n}^{m} a_k| \leq \sum_{k=n}^{m} |a_k| < \epsilon \ \forall\ n, m > N_\epsilon$. □

The other way does not necessarily hold. (There can exist series that converge, but do not converge if you replace the summands by their absolute values. We will discuss absolute vs conditional convergence later.)

The next theorem is the basis for most convergence tests. (The comparison test.)

Theorem 5. *1. If $|a_n| \leq b_n \ \forall \ n \geq N$ for a fixed N where $\sum b_n$ converges, then so does $\sum |a_n|$ (and hence $\sum a_n$).*

2. If $a_n \geq d_n \geq 0 \ \forall \ n \geq N$ for a fixed N, and $\sum d_n$ diverges, then so does $\sum a_n$ diverge.

Proof. 1. Indeed, $\sum_{k=n}^{m} |a_k| \leq \sum_{k=n}^{m} b_k < \epsilon \ \forall n, m > N_\epsilon$.

2. By monotonicity, $\sum d_n$ is not bounded above. Therefore, given $M > 0$ there exists an N so that $\sum_{k=1}^{N} d_k > M$. This means that $\sum_{k=1}^{N} a_k > M$. Thus $\sum a_n$ diverges.

□

6 Series of non-negative terms

The first famous example is that of the geometric series :
If $x \geq 1$ *or* $x \leq -1$ then $1 + x + x^2 + \ldots$ diverges. Otherwise it converges to $\frac{1}{1-x}$.