Notes for 2 Mar (Thursday)

1 Recap

1. Defined uniform continuity and proved that continuous functions on compact sets are uniformly continuous.

2. Proved that continuous functions take connected sets to connected sets. Derived the Intermediate value theorem as a corollary.

3. Defined limits at infinity, classified discontinuities, and started proving that monotonic functions have only jump discontinuities.

2 Assorted topics in continuity

Theorem 1. *If $f : (a,b) \to \mathbb{R}$ is monotonically increasing, then $f(x+)$ and $f(x-)$ exist for every point $x \in (a,b)$. Moreover, $f(x+) \leq f(y-)$ for $x < y$. Furthermore, $f(x+) = \inf_{b>t>x} f(t)$ and $f(x-) = \sup_{a<t<x} f(t)$.*

Proof. On (x,b), $f(t)$ is bounded below by $f(x)$. Thus $M = \inf_{b>t>x} f(t)$ exists. Likewise, $m = \sup_{a<t<x} f(t)$ exists. For every $\epsilon > 0$, $M + \epsilon > f(t) \geq M$ for some $b > t > x$. Since f is increasing, $M + \epsilon > f(y) \geq M \ \forall\ t \geq y > x$. Suppose $y_n > x \to x$. Then $M + \epsilon \geq \limsup f(y_n) \geq \liminf f(y_n) \geq M$. Since this is true for all ϵ, $\lim f(y_n) = M$. Since this is true for all sequences $y_n > x$, this means that $f(y_n+) = M$. Likewise $f(x-) = m$. Moreover, suppose $x < y$. Then for all $y > y_1 > t > x$, $f(t) \leq f(y_1) \leq f(y)$. Since $f(x+)$ is also equal to (by monotonicity of f) $\inf_{y_1>t>x} f(t)$ we see that $f(x+) \leq f(y_1)$. Therefore, $f(x+) \leq f(y-)$. $\square$

We can use the above result to prove that monotonic functions have at most countably many discontinuities.

Theorem 2. *Let f be monotonic on (a,b). Then it has at most countably many discontinuities.*

Proof. Assume without loss of generality that f is monotonically increasing. (If not, $-f$ is monotonically increasing and both have the same sets of discontinuities.) Let E be the set of discontinuities of f. Note that all of these are jump discontinuities. Moreover, $f(x-) \leq f(x) \leq f(x+)$ for all x by monotonicity. Thus the only way to have a discontinuity at x is if $f(x+) - f(x-) > 0$. Note that there is no $t \in (a,b)$ such that $f(t) \in (f(x-), f(x+))$. Choose a rational $q_x \in (f(x-), f(x+))$. Therefore, for each

$x \in E$ we have chosen a rational q_x. Now if $q_x = q_y$ then surely $x = y$. Therefore, there is an injection from E to a countable set. Thus E is at most countable. (Either it is finite or an infinite subset of a countable set.) $\square$

This however does not mean that the set of discontinuities of f are isolated. In fact, given *any* countable set $E \subset (a, b)$ we can construct a monotonic function having discontinuities exactly on E. Indeed, suppose the elements of E are enumerated as $x_1, x_2, \ldots$. Let c_n be any sequence of positive reals such that $\sum c_n$ converges. Define $f(x) = 0 + \sum_{n \text{ such that } x_n < x} c_n$ where $a < x < b$. Then we claim that

1. f makes sense.
   Indeed, since c_n is absolutely convergent, the order of summation does not matter as far as convergence is concerned.

2. It is monotonically increasing.
   If $x < y$, then surely $\{n \text{ such that } x_n < x\} \subset \{n \text{ such that } x_n < y\}$. Thus $f(x) \leq f(y)$.

3. $f(x_n+) - f(x_n-) = c_n$.
   Suppose $y_m \to x_n$ such that $y_m < x_n$ and $z_m \to x$ such that $z_m > x_n$. Note that $\lim f(z_m) = f(x_n+)$ and $\lim f(y_m) = f(x_n-)$. $f(z_m) \geq c_n + f(y_m)$. Thus $f(x_n+) - f(x_n-) \geq c_n$. Suppose $f(x_n+) - f(x_n-) > c_n$. Then certainly there is some M such that for all $m > M$, $f(z_m) - f(y_m) > c_n + \epsilon$ for some fixed small enough $\epsilon > 0$. This means that $\sum_{l \textit{ such that } x_n < x_l < z_m} c_l > \epsilon$ for all $m > M$. But by absolute convergence, surely there is an m such that the left-hand side is $< \epsilon$. Indeed, reorder the natural numbers by saying that $n\text{``}<\text{''}m$ if $x_n < x_m$. Therefore for all $l\text{``}>\text{''}L\text{``}>\text{''}n \sum c_l < \epsilon$. Choose $m$ to be so large there are no $x_i \in (x_n, z_m)$ such that $i\text{``}<\text{''}L$. This is a contradiction.

4. Everywhere else it is continuous.
   Suppose $x \notin E$. Let $y_n \to x$ such that $y_n < x$ and $z_n \to x$ such that $z_n > x$. Suppose $f(x+) > f(x-)$. Fix $\epsilon < f(x+) - f(x-)$. The same argument as before produces a contradiction.

3 Differentiability

Here is a familiar definition : A function $f : (a, b) \to \mathbb{R}$ is said to be differentiable at x with derivative $f^{'}(x)$ if

$$\lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = f^{'}(x) \tag{1}$$

which is equivalent to

$$\lim_{y \to x} \frac{f(y) - f(x)}{y - x} = f^{'}(x) \tag{2}$$

One can potentially define left and right derivatives at the endpoints but it gives only a misleading sense of comfort. In higher dimensions such notions become tricky to handle. So might as well consider differentiabilitiy only on an open set right from the beginning.

Theorem 3. *If f is differentiable at x then it is continuous at x.*

Proof.

$$\lim_{y\to x}(f(y)-f(x)) = \lim_{y\to x}\frac{f(y)-f(x)}{y-x}(y-x) = f^{'}(x)\times 0 = 0 \tag{3}$$

□

Then we have the usual sum, product and quotient rules of differentiation.

Theorem 4. *If $f^{'}$, $g^{'}$ exist, then $f+g$, αf, and fg are differentiable. If $g^{'}\neq 0$ then so is f/g. Also, $(f+g)^{'} = f^{'}+g^{'}$, $(\alpha f)^{'} = \alpha f^{'}$, $(fg)^{'} = f^{'}g + fg^{'}$ and $(f/g)^{'} = \frac{f^{'}g-g^{'}f}{g^2}$.*

Proof. 1. The first one follows from the sum rule of limits.

2.

$$\begin{aligned}\lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x)g(x)}{h} &= \lim_{h\to 0}\frac{(f(x+h)-f(x))g(x+h)+f(x)(g(x+h)-g(x))}{h}\\ &= f^{'}g+g^{'}f\end{aligned} \tag{4}$$

3. The quotient rule has a similar proof relying on limit laws.

□

The next order of business is the infamous chain rule.

Theorem 5. *Suppose $f:(a,b)\to\mathbb{R}$ is differentiable at a point x, $g:E\to\mathbb{R}$ (where E contains the image of f) is differentiable at $f(x)$, then $h = g\circ f$ is differentiable at x with derivative $h^{'}(x) = g^{'}(f(x))f^{'}(x)$.*

Proof.

$$\lim_{h\to 0}\frac{g(f(x+h))-g(f(x))}{h} = \lim_{h\to 0}\frac{g(f(x+h))-g(f(x))}{f(x+h)-f(x)}\frac{f(x+h)-f(x)}{h}$$

If $f(x+h)\neq f(x)$ for all $|h|<\delta$ for some δ, since f is continuous at x then $f(x+h)-f(x) = y$ goes to 0 as $h\to 0$. Thus the first limit is $g^{'}(f(x))$ because g is differentiable at $f(x)$ and the second one is $f^{'}(x)$.
But we do not have this assumption. So we have to proceed in a different way which will eventually prove to be useful in multivariable calculus. Let $y_0 = f(x)$. Since g is differentiable at y_0, we see that

$$|g(y)-g(y_0)-g^{'}(y_0)(y-y_0)|\leq\epsilon|y-y_0| \tag{5}$$

for all $|y - y_0| < \delta$. Dividing by $|h|$ on both sides,

$$|\frac{g(y) - g(y_0)}{h} - g^{'}(y_0)\frac{y - y_0}{h}| \leq \epsilon|\frac{y - y_0}{h}| \tag{6}$$

Choose h so small that $y_h = f(x + h)$ satisfies $|y_h - y_0| < \delta$. Take any sequence $h_n \to 0$. Taking lim sup on both sides of 6 we see that

$$\begin{aligned}
\lim_{n\to\infty} g^{'}(y_0)\frac{y_n - y_0}{h_n} - \epsilon \limsup |\frac{y_n - y_0}{h_n}| &\leq \limsup \frac{g(y_n) - g(y_0)}{h_n} \\
&\leq \lim_{n\to\infty} g^{'}(y_0)\frac{y_n - y_0}{h_n} + \epsilon \limsup |\frac{y_n - y_0}{h_n}| \\
g^{'}(y_0)f^{'}(x) - \epsilon f^{'}(x) \leq \limsup \frac{g(y_n) - g(y_0)}{h_n} &\leq g^{'}(y_0)f^{'}(x) + \epsilon f^{'}(x) \qquad (7)
\end{aligned}$$

Taking $\epsilon \to 0$ we see that $\limsup \frac{g(y_n)-g(y_0)}{h_n} = g^{'}(y_0)f^{'}(x)$. The same thing applies to lim inf. Therefore the limit is indeed $g^{'}(y_0)f^{'}(x)$ for any sequence. Thus the chain rule holds. □

Here is an interesting example : $f(x) = x^2 \sin(1/x)$ when $x \neq 0$ and $f(0) = 0$. Note that

$$\lim_{h\to 0} \frac{f(h) - f(0)}{h} = \lim_{h\to 0} h \sin(\frac{1}{h}) = 0 \tag{8}$$

by the squeeze rule. (Notice that although I have not defined $\sin(x)$ yet, whatever it is, we just need to know that it is bounded to conclude this.) Once we define trigonometric functions, we will see that when $x \neq 0$, $f^{'}(x) = 2x \sin(1/x) - \cos(1/x)$. Unfortunately the limit as $x \to 0$ does not exist. So this function is differentiable at $x = 0$ but $f^{'}$ is not continuous there.

4 Mean value theorems

Definition : If $f : X \to \mathbb{R}$ is a function, then f has a local maximum at $p \in X$ if there exists a $\delta > 0$ such that $f(q) \leq f(p)$ for all $q \in B_\delta(p)$. Likewise for local minima.

Theorem 6. *Let $f : [a, b] \to \mathbb{R}$ be a function. If it has a local maximum (or minimum) at $x \in (a, b)$ and if $f^{'}$ exists, then $f^{'}(x) = 0$.*

Proof. Suppose $h_n > 0$ is a sequence tending to 0. Then $f^{'}(x) = \lim_{n\to\infty} \frac{f(x+h_n)-f(x)}{h_n} \leq 0$. Likewise, if $h_n < 0$ is a sequence tending to 0, then $f^{'}(x) \geq 0$. Hence $f^{'}(x) = 0$. □

We have Rolle's theorem.

Theorem 7. *If $f : [a, b] \to \mathbb{R}$ is continuous, $f(a) = f(b)$, and differentiable on (a, b), then there exists a $c \in (a, b)$ such that $f^{'}(c) = 0$.*

Proof. Since f is continuous, it achieves a maximum and a minimum somewhere. If both occur on the endpoints, then f is a constant and we are done. If one of them occurs in the interior at a point c, then by the previous theorem, $f^{'}(c) = 0$. □

As a corollary we have Cauchy's mean value theorem.

Theorem 8. *If f and g are continuous real functions on $[a, b]$ that are differentiable in (a, b) then there exists a $c \in (a, b)$ such that $[f(b) - f(a)]g'(c) = [g(b) - g(a)]f'(c)$.*

Proof. We have to use Rolle's theorem somehow. Clearly the right function to choose is $h(x) = [f(b) - f(a)]g(x) - [g(b) - g(a)]f(x)$. It satisfies $h(a) = h(b)$. Therefore there is a $c \in (a, b)$ such that $h'(c) = 0$. $\square$

As a consequence, if $f' \geq 0$ throughout (a, b) then f is increasing. (Use the Cauchy mean value theorem with $g(x) = x$.) Likewise for decreasing.