Notes for 31 Mar (Friday)

1 Recap

1. Motivated and stated the Weierstrass approximation theorem : Continuous functions on $[a, b]$ can be uniformly approximated by polynomials.

2. Proved the theorem in a constructive manner using probability. Essentially, we said that if you throw a biased coin a large number of times, you can write the average amount of money you receive in two different ways. We compared the two ways.

3. Started proving it in Rudin's way. The idea is to approximate the Dirac delta by polynomials. So we chose $Q_n(x) = c_n(1 - x^2)^n$.

2 The Weierstrass approximation theorem

Proof. (*Cont'd from the last time...*)
Let us analyse $Q_n(t)$ when t is small and when t is large. For this, we need to know how c_n behaves. Note that $g(x) = (1 - x^2)^n - 1 + nx^2 \geq 0$ when $0 \leq x \leq 1$.

$$\int_{-1}^{1}(1-x^2)^n dx = 2\int_{0}^{1}(1-x^2)^n dx > 2\int_{0}^{1/\sqrt{n}}(1-x^2)^n dx$$
$$> 2\int_{0}^{1/\sqrt{n}}(1-nx^2)dx = 2(\frac{1}{\sqrt{n}} - \frac{n}{3n\sqrt{n}}) > \frac{1}{\sqrt{n}}$$

Thus $c_n < \sqrt{n}$. Thus on $\delta \leq x \leq 1$ where $\delta > 0$, $Q_n(x) \leq \sqrt{n}(1 - \delta^2)^n$ which goes to 0. Therefore $Q_n(x)$ goes to 0 uniformly on $[\delta, 1]$.

Returning back to $P_n - f$ we see that by uniform continuity of f we can choose a δ so that $|f(x + t) - f(x)| < \frac{\epsilon}{2}$ if $|t| < \delta$. Now

$$|P_n(x) - f(x)| < \int_{-\delta}^{\delta} Q_n(t)\frac{\epsilon}{2}dt + \int_{|t|\geq\delta} Q_n(t)2M dt$$
$$< \frac{\epsilon}{2} + 2M\sqrt{n}(1-\delta^2)^n < \epsilon \qquad (1)$$

if $n > N$. □

In particular, we have the following corollary : For every interval $[-a, a]$ there is a sequence of real polynomials P_n such that $P_n(0) = 0$ converging to $|x|$ uniformly on $[-a, a]$. Indeed, by Weierstrass there exist $\tilde{P}_n$ converging uniformly to $|x|$. Thus

$P_n = \tilde{P}_n(x) - \tilde{P}_n(0)$ do the job.

The Weierstrass approximation theorem can be used to prove a generalisation called the Stone-Weierstrass theorem. Since it is too abstract, we will skip it. But it implies that Weierstrass approximation holds in arbitrary number of variables.

3 Power series

A function $f : (a-R, a+R) \to \mathbb{R}$ is called analytic if $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$ where the series converges absolutely, i.e., f is a power series. We can calculate the radius of convergence, etc (using the Ratio test and so on from another era of this class).
The nice thing about these power series or analytic functions is that, as long as you are within the radius of convergence, you can differentiate and integrate term-by-term. Indeed,

Theorem 1. *Suppose $\sum_{n=0}^{\infty} c_n x^n$ converges for $|x| < R$ and define $f(x) = \sum_{n=0}^{\infty} c_n x^n$. This series converges to f uniformly on $[-R+\epsilon, R-\epsilon]$ for every ϵ. This function is continuous and differentiable in $(-R, R)$ and $f^{'}(x) = \int_{n=1}^{\infty} n c_n x^{n-1}$ when $x \in (-R, R)$. Moreover, $\int_a^b f(x)dx = \sum \frac{c_n}{n+1}(b^{n+1} - a^{n+1})$ where $a, b \in (-R, R)$.*

Proof. Recall that every power series converges absolutely within the radius of convergence (Ratio test). Now when $|x| \leq R - \epsilon$, $c_n x^n \leq |c_n (R-\epsilon)^n$. Since $\sum c_n x^n$ converges on $(-R, R)$, this means that $R \leq$ the radius of convergence. Hence $\sum c_n (R-\epsilon)^n$ converges. By the Weierstrass M-test, this means that $\sum c_n x^n$ converges uniformly to f on $[-R+\epsilon, R-\epsilon]$. Thus f is continuous. Now let's compute the radius of convergence of $\sum n c_n x^{n-1}$. Indeed, this converges if $\limsup \frac{(n+1)|c_{n+1}|}{n|c_n|}|x| < 1$, i.e., the radius of convergence is $\frac{1}{\limsup((n+1)|c_{n+1})/(n|c_n|)}$ which I claim is equal to that of the original series. Indeed, this can be done either using the definition or using subsequences. Therefore $\sum n c_n x^{n-1}$ converges uniformly to a function $g(x)$ on $[-R+\epsilon, R+\epsilon]$. By a theorem we proved earlier, indeed $g = f^{'}$. Since we can interchange uniform limits and RS integrals, the statement for integration follows. □

As a corollary f has derivatives of all orders in $(-R, R)$ given by differentiating term-by-term.

Let's prove Abel's theorem on the product of series, i.e., if $\sum a_n = A$, $\sum b_n = B$, and if $\sum c_n = C$ exist where c_n is the Cauchy product of a_n, b_n then $C = AB$. To do so, we consider $f(x) = \sum a_n x^n$, $g(x) = \sum b_n x^n$, and $h(x) = \sum c_n x^n$ on $[0, 1]$. Of course they all converge because A, B, C exist (which are $f(1), g(1), h(1)$) and we know that power series converge on intervals. So the radius of convergence is greater than or equal to 1. Note that for $x < 1$ these series converge absolutely and hence $f(x)g(x) = h(x)$. If we manage to take the limit as $x \to 1$ of $f(x), g(x), h(x)$ and prove that indeed the limit is what we expect, then we will be done. This we prove here :

Theorem 2. *Suppose* $\sum c_n$ *converges. Put* $f(x) = \sum_{n=0}^{\infty} c_n x^n$ *when* $-1 < x < 1$*. Then* $\lim_{x \to 1^-} f(x) = \sum c_n$*.*