Notes for 3 Feb (Friday)

1 A longish recap

1. Defined monotonic sequences and showed that bounded monotonic sequences converge (and vice-versa).
2. Defined extended reals, lim sup, and lim inf. Recall that given a sequence, the supremum of all subsequential limits is called lim sup. It is also characterised as $\lim_{N\to\infty} \sup_{n\geq N} x_n$.
3. Calculated the limits of some standard sequences.
4. Defined series, and spoke about the divergence and the comparison tests. Also paid lip service to the geometric series.

Let us calculate some examples of lim sup and lim inf so that we understand them better.

1. The lim sup of $1, -1, 1, -1, \ldots$ is 1 and the lim inf is -1. (Why? You can do it using the definition if you want. The only subsequential limits are ± 1.)
2. What is are the lim sup and lim inf of $a_n = n^{\sin(n\pi/2)}$? The sequence $\sin(n\pi/2)$ is $1, 0, -1, 0, 1, 0, -1, \ldots$. So a_n is $1, 2^0, 3^{-1}, 4^0, 5^1, 6^0, \ldots$. In other words, the subsequence $1, 5, 9, \ldots$ converges to ∞. Therefore, $\limsup$ is infinity. (After all, lim sup is the maximum of all subsequential limits.) All the terms of this sequence are positive. Therefore any subsequential limit is ≥ 0. Now the subsequence $1, \frac{1}{3}, \frac{1}{7}, \ldots$ converges to 0. Thus $\liminf a_n = 0$.
3. Find the lim sup and lim inf of $x_n = \sin(n\pi/3)$. So the sequence is $\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}, 0, -\frac{\sqrt{3}}{2}, -\frac{\sqrt{3}}{2}, 0, \ldots$. For the sake of variety let's do this using the other characterisation of lim sup and lim inf.

$$\begin{aligned}
\limsup a_n &= \lim_{N\to\infty} \sup_{n\geq N} \sin(n\pi/3) \\
&= \lim_{N\to\infty} \frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{2} \\
\liminf a_n &= \lim_{N\to\infty} \inf_{n\geq N} \sin(n\pi/3) \\
&= \lim_{N\to\infty} -\frac{\sqrt{3}}{2} = -\frac{\sqrt{3}}{2}
\end{aligned}$$

4. Suppose a_n and b_n are two series such that $b_n \to b$. Then $\limsup(a_n + b_n) = \limsup a_n + b$ (and likewise for lim inf) -
   Indeed, suppose we take a convergent subsequence $a_{n_1}+b_{n_1}, a_{n_2}+b_{n_2}, \ldots$, then indeed it converges to $\lim_{k\to\infty} a_{n_k} + b$ (Note that if $C_n = A_n + B_n$ and B_n converge, then so does A_n.) Therefore the supremum of all subsequential limits is $\limsup a_n + b$.

Another small, subtle point : Of course we know that a sequence converges if and only if all its subsequences converge to the same limit. However, what if we know that all of its *convergent* subsequences converge to the same limit ? Can we still conclude that the sequence converges ? (i.e., that there are no divergent subsequences ?) The answer is yes. It is provided by the following lemma.

Lemma 1.1. *If every subsequence x_{n_i} of x_n has* a *subsequence $x_{n_{i_j}}$ that converges to the same limit L, then x_n itself converges to L.*

Proof. Suppose not. That is, there exists an $\epsilon > 0$ such that for every natural N, there is a $n_N > N$ so that $|x_{n_N} - L| \geq \epsilon$. Now the subsequence x_{n_i} has a convergent subsequence that converges to L. But this is a contradiction. □

Why does this imply what we want ? This is because every subsequence does have a convergent subsequence (convering to its lim sup for example). Since we assumed that every convergent subsequence converges to the same limit, by the previous lemma, this means that the original sequence converges (i.e. there are no divergent subsequences).

2 Non-negative series

What is interesting is the following theorem of Cauchy. (Don't know how or why he came up with it but it is surprisingly useful.)

Theorem 1. *Suppose $a_1 \geq a_2 \geq \ldots \geq 0$. Then $\sum_{n=1}^{\infty} a_n$ converges if and only if $\sum_{n=0}^{\infty} 2^n a_{2^n} = a_1 + 2a_2 + 4a_4 + \ldots$ does.*

Proof. 1. Assume that $\sum_{n=0}^{\infty} 2^n a_{2^n} = a_1 + 2a_2 + 4a_4 + \ldots$ converges.

First we claim that $\sum_{k=0}^{m} 2^k a_{2^k} \geq \sum_{k=1}^{2^{m+1}-1} a_k$. This can be proven by induction. Indeed, this is true for $m = 0$. Suppose it is true for m, then for $m + 1$,

$$\sum_{k=0}^{m+1} 2^k a_{2^k} = \sum_{k=0}^{m} 2^k a_{2^k} + 2^{m+1} a_{2^{m+1}}$$

$$\geq \sum_{k=1}^{2^{m+1}-1} a_k + a_{2^{m+1}} + a_{2^{m+1}} + \ldots \geq \sum_{k=1}^{2^{m+1}-1} a_k + a_{2^{m+1}} + a_{2^{m+1}+1} + \ldots + a_{2^{m+2}-1} = \sum_{k=1}^{2^{m+2}-1} a_k. \tag{1}$$

Thus since $\sum_{n=0}^{\infty} 2^n a_{2^n}$ is bounded, so is $\sum_{n=1}^{\infty} a_n$ and by non-negativity, it converges.

2. Assume that $\sum_{n=0}^{\infty} 2^n a_{2^n} = a_1 + 2a_2 + 4a_4 + \ldots$ diverges.
   Indeed, note that $a_1 + 2a_2 \leq 2a_1 + 2a_2$, $4a_4 \leq 2(a_3 + a_4)$, $8a_8 \leq 2(a_5 + a_6 + a_7 + a_8)$ and so on. By induction, $2^m a_{2^m} \leq 2(a_{2^{m-1}+1} + \ldots + a_{2^m})$. This along with the monotonicity/non-negativity thing proves the result.

$\square$

The first application is towards the so-called p-series. (The Harmonic series is a special case of this.)

Theorem 2. $\sum \frac{1}{n^p}$ *converges when* $p > 1$ *and diverges when* $p \leq 1$.

Proof. Indeed, applying Cauchy's theorem, we have to check $\sum \frac{2^n}{2^{np}} = \sum \frac{1}{2^{n(p-1)}}$ from which the result follows using the geometric series. $\square$

Assuming the existence of the natural log function (which we will discuss later anyway and this theorem will not be used anywhere else in life, so it is not circular),

Theorem 3. $\sum \frac{1}{n(\ln n)^p}$ *converges when* $p > 1$ *and diverges when* $p \leq 1$.

Proof. Indeed, applying Cauchy's theorem, we have to check $\sum \frac{2^n}{2^n(\ln 2^n)^p} = \sum \frac{1}{(\ln 2)^p n^p}$ which reduces it to the convergence of the p-series. $\square$

3 The number e

Definition : $e = \sum_{n=0}^{\infty} \frac{1}{n!}$. How do we know it converges ? Well $n! \geq 2^{n-1}\ \forall\ n \geq 1$. Use the comparison test comparing with a geometric series. That also shows that $e < 3$. Of course $e > 2$.

But very commonly, it is defined in another way.

Theorem 4.

$$e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n \tag{2}$$

Proof. The only tool we have at our disposal is the binomial theorem. We don't even know if the right hand side converges. Fortunately, we have another tool at our disposal - lim sup and lim inf.

Let $s_n = \sum_{k=0}^{n} \frac{1}{k!}$ and $t_n = \sum_{k=0}^{n} \binom{n}{k} \frac{1}{n^k} = \sum_{k=0}^{n} \frac{1}{k!}(1 - \frac{1}{n})(1 - \frac{2}{n}) \ldots (1 - \frac{k-1}{n})$.

Now intuitively, as $n \to \infty$ it seems reasonable that t_n gets close to s_n. At least it is obvious that $t_n \leq s_n\ \forall\ n$. So we want to say $\lim t_n \leq e$ but we do not know whether the limit exists. Fortunately, we can say $\limsup t_n \leq e$. If we just prove that $\liminf t_n \geq e$

we will be done.
What we really feel like doing is just take the goddamned limit as $n \to \infty$ of the pesky factors like $(1 - \frac{1}{n})$ without affecting the $\frac{1}{k!}$ terms. There is a nice trick to do this. Namely, if $n \geq m$, then $t_n \geq \sum_{k=0}^{m} \frac{1}{k!}(1-\frac{1}{n})(1-\frac{2}{n})\dots(1-\frac{k-1}{n})$. Now we may take the lim inf as $n \to \infty$ whilst keeping m fixed. Therefore, $\liminf t_n \geq \sum_{k=0}^{m} \frac{1}{k!} \ \forall \ m$. Hence $\liminf t_n \geq e$. □

Here is another important result :

Theorem 5. *The number e is irrational.*

Proof. Suppose $e = \frac{p}{q}$. An easy geometric series comparison shows that $0 < e - s_n < \frac{1}{n!n}$. Thus $0 < q!(e - s_q) < \frac{1}{q}$. But the difference of two integers cannot be less than a fraction and yet be positive. □

Actually e is not even algebraic, i.e., there is no polynomial whose coefficients are rational numbers such that e is a root of such a polynomial. (Notice that we can prove that algebraic numbers are countable, meaning that there are lots of non-algebraic numbers. But this is our first example. They are actually hard to come by!) But this is not easy to prove. As a consequence, e^2 is not rational, or for that matter e^{1000} is not rational.

4 Ratio and root tests

Given $\sum a_n$ let $L = \limsup |a_n|^{1/n}$.

Theorem 6. *If*

1. *$L < 1$ the series converges.*
2. *$L > 1$ it diverges.*
3. *$L = 1$, pray because this test tells you nothing.*

Proof. The idea is to compare with the geometric series. Roughly speaking, when n is large, the series roughly looks like a geometric series with the common ratio L.

Indeed, suppose $L < 1$. Then $L + \epsilon < 1$ for a small $\epsilon > 0$. By the properties of lim sup we know that there exists an N so that $|a_n|^{1/n} < L + \epsilon \ \forall \ n \geq N$. This means that $|a_n| \leq (L+\epsilon)^n \ \forall \ n \geq N$. By the comparison test this means that $\sum |a_n|$ converges.

If $L > 1$ and finite (the infinity case is trivial), then $L - \epsilon > 1$ for a small $\epsilon > 0$. Since $L = \lim_{N\to\infty} \sup_{n\geq N} |a_n|^{1/n}$ we see that for a large N the quantity $|a_n|^{1/n} > L - \epsilon \ \forall \ n > N$. Therefore $|a_n| > (L-\epsilon)^n \ \forall \ n > N$. Thus by the comparison test, the series diverges. □

Likewise, here is the ratio test.

Theorem 7. *Let $\alpha = \limsup \frac{|a_{n+1}|}{|a_n|}$. (Assume that none of the a_n are 0.) If*

1. *If $\alpha < 1$ the series converges.*

2. *If* $\frac{|a_{n+1}|}{|a_n|} \geq 1$ *for all* $n \geq N$ *then it diverges.*

Proof. The philosophy is exactly the same as the previous proof. If $\alpha < 1$, then so is $\alpha + \epsilon < 1$ for a small $\epsilon > 0$. By properties of lim sup we see that $\frac{|a_{n+1}|}{|a_n|} < \alpha + \epsilon \ \forall\ n \geq N$. Thus $|a_m| \leq \alpha + \epsilon)^{m-N}|a_N| \ \forall\ m \geq N$. By the comparison test the series converges.

Likewise, if $\frac{|a_{n+1}|}{|a_n|} \geq 1$ for all large $n \geq N$, then $|a_m| \geq a_N$ and hence $\lim_{m\to\infty} |a_m| \neq 0$. This shows (by the divergence test) that the series diverges. □

The root test is slightly more powerful than the ratio test (although I always prefer to use the ratio test). Indeed,

Theorem 8.

$$\liminf \frac{|a_{n+1}|}{|a_n|} \leq \liminf |a_n|^{1/n} \leq \limsup |a_n|^{1/n} \leq \limsup \frac{|a_{n+1}|}{|a_n|} \tag{3}$$

Proof. We will prove the lim sup inequality. (The lim inf case is similar.) Firstly note that $\frac{|a_n|}{|a_N|} = \frac{|a_n|}{|a_{n-1}|}\frac{|a_{n-1}|}{|a_{n-2}|}\ldots$. Suppose $L = \limsup \frac{|a_{n+1}|}{|a_n|}$. If $L = \infty$ then it is trivial. Assume L to be finite. Then by properties of lim sup, $L + \epsilon > \frac{|a_n|}{|a_{n-1}|} \ \forall\ n > N_\epsilon$. Therefore, $|a_n|^{1/(n-N_\epsilon)} < |a_{N_\epsilon}|^{1/(n-N_\epsilon)}(L+\epsilon) \ \forall\ n > N_\epsilon$. Thus

$$|a_n|^{1/n} < |a_{N_\epsilon}|^{1/n}(L+\epsilon)^{(n-N_\epsilon)/n}$$

Taking lim sup on both sides and using a standard limit we see that $\limsup |a_n|^{1/n} \leq L + \epsilon \ \forall\ \epsilon > 0$. This allows us to conclude the result. □